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STATE SPACES OF C^* -ALGEBRAS

by

Erik M. Alfsen
Oslo

Frederic W. Shultz
Wellesley College

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§ 1. Introduction

The purpose of this paper is to characterize the state spaces of C^* -algebras among the state spaces of all JB-algebras. In a previous paper [6] we have characterized the state spaces of JB-algebras among all compact convex sets. Together, these two papers give a complete geometric characterization of the state spaces of C^* -algebras.

Recall from [6] that the state spaces of JB-algebras will enjoy the Hilbert ball property, by which the face $B(\rho, \sigma)$ generated by an arbitrary pair ρ, σ of extreme states is (affinely isomorphic to) the unit ball of some real Hilbert space, and that there actually exist such faces of any given (finite or infinite) dimension for suitably chosen JB-algebras. In the present paper

we show that for an arbitrary pair ρ, σ of extreme states of a C^* -algebra, then the dimension of $B(\rho, \sigma)$ is three or one, with the latter being some sort of a degeneracy (Proposition 3.3). This statement, which we term the 3-ball property, is the first of our axioms for state spaces of C^* -algebras. The second and last axiom is a requirement of orientability: the state space K of a JB-algebra with the 3-ball property is said to be orientable if it is possible to make a "consistent" choice of orientations for the 3-balls $B(\rho, \sigma)$ in the w^* -compact convex set K , the idea being that the orientation shall never be suddenly reversed by passage from one such ball to a neighbouring one. (See § 6 for the precise definition). Thus we have the following:

Main Theorem. A JB-algebra A with state space K is (isomorphic to) the self-adjoint part of a C^* -algebra iff K has the 3-ball property and is orientable.

Note that a C^* -algebra, unlike a JB-algebra, is not completely determined by the affine geometry and the w^* -topology of its state space. However, the state space does determine the Jordan structure, and with this prescribed we have a 1-1 correspondence between C^* -structures and consistent orientations of the state space (Corollary 7.3). Thus, for C^* -algebras the oriented state space is a dual object from which we can recapture all relevant structure.

We will now briefly discuss the background for the problem, and then indicate the content of the various sections.

By results of Kadison [24], [26], [29], the self-adjoint part \mathcal{A}_{sa} of a C^* -algebra \mathcal{A} with state space K is isometrically

order-isomorphic to the space $A(K)$ of all w^* -continuous affine functions on K . More specifically, \mathcal{O}_{sa} is an order unit space (a "function system" in Kadison's terminology), and the order unit spaces A are precisely the $A(K)$ -spaces where K is a compact convex subset of a locally convex Hausdorff space; (in fact K can be taken to be the state space of A , formally defined as in the case of a C^* -algebra). Thus, the problem of characterizing the state spaces of C^* -algebras among all compact convex sets, is equivalent to that of characterizing the self-adjoint parts of C^* -algebras among all order unit spaces. This problem is of interest in its own right, and it also gains importance by the applications to quantum mechanics, where the order unit space \mathcal{O}_{sa} represents bounded observables, while the full C^* -algebra \mathcal{O} is devoid of any direct physical interpretation. Note in this connection that the Jordan product in \mathcal{O}_{sa} (unlike the ordinary product in \mathcal{O}) is physically relevant, and that the pioneering work on Jordan algebras by Jordan, von Neumann and Wigner [19] was intended to provide a new algebraic formalism for quantum mechanics (cf. also [30]).

In [25] Kadison proved that the Jordan structure in the order unit space \mathcal{O}_{sa} is completely determined, in that any unital order automorphism of \mathcal{O}_{sa} is a Jordan automorphism, and he pointed out the great importance of the Jordan structure for the study of C^* -algebras. The investigation of more general Jordan algebras was continued in a series of papers by Topping, Størmer and Effros [39], [36], [37], [18]. Their approach was more general than that of Jordan, von Neumann and Wigner in that they considered infinite dimensional algebras, but it was less general in that their algebras were assumed a priori to be algebras of bounded self-adjoint operators on a Hilbert space ("JC-algebras" in Topping's terminology). A non-

spatial investigation of normed Jordan algebras was carried out in [7]. Here the basic notion is that of a JB-algebra, which is defined to be a real Jordan algebra with unit 1 which is also a Banach space, and where the Jordan product and the norm are related as follows:

$$(1.1) \quad \|a \circ b\| \leq \|a\| \|b\|, \quad \|a^2\| = \|a\|^2, \quad \|a^2\| \leq \|a^2 + b^2\|.$$

These axioms are closely related to those of Segal [32], and the JB-algebras will include the finite dimensional formally real algebras studied by Jordan, von Neumann and Wigner (which can be normed in a natural way), as well as the Jordan operator algebras studied by Topping, Størmer and Effros. The main result of [7] states that the study of general JB-algebras can be reduced to the study of Jordan operator algebras and the exceptional algebra M_3^8 of all self-adjoint 3×3 -matrices over the Cayley numbers. (For related results, see [34]).

Turning to the geometry of the state space K for a C^* -algebra \mathcal{A} , we have a close relationship between the facial structure of K and the ideal structure of \mathcal{A} . This relationship was recognized independently by Effros [17] and Prosser [31]. They showed that there is a 1-1 correspondence between the norm closed (respectively w^* -closed) faces of K and the ultra-weakly closed one-sided ideals in the enveloping von Neumann algebra \mathcal{A}^{**} (respectively the norm closed ideals in the given C^* -algebra \mathcal{A}), and that the latter in turn are in 1-1 correspondence with the projections in \mathcal{A}^{**} (respectively the upper semi continuous projections in \mathcal{A}^{**}). Note that this already gives some insight in the geometry of state spaces. Thus, while the state space of M_2 (the 2×2 -matrix algebra over \mathbb{C}) is known to be a Euclidean 3-ball

(cf. e.g. [4 ; p.103]), it is now seen that the state spaces of M_n for $n \geq 3$ are not strictly convex since they contain non-trivial faces corresponding to the one sided ideals $J \cong M_2, \dots, M_{n-1}$.

Effros and Prosser also showed that the above correspondence between ideals and projections has a two-sided counterpart; specifically, one may replace the word "one-sided" by "two-sided" if at the same time the term "projection" is replaced by "central projection". To complete the list of correspondences, we must have the simple, but important, notion of a split face which was introduced in [2]. (A face F of a convex set K is "split" if there is another, necessarily unique, face F' such that K is the direct convex sum of F and F'). Now the replacement of the term "one-sided" by "two-sided" corresponds to a replacement of the term "face" by "split face". In particular, the w^* -closed split faces of the state space K of a C^* -algebra \mathcal{O} are the annihilators (in K) of the norm closed two-sided ideals in \mathcal{O} . This was the starting point for the investigation of w^* -closed split faces of compact convex sets in [2]; here the main result is a dominated extension theorem for real valued affine functions, which was later generalized to Banach space valued functions by Andersen [8], and then applied by Andersen [9] and Vesterstrøm [41] to provide lifting theorems with applications to non-commutative cohomology.

The (norm- and w^* -) closed faces of the state space K of a C^* -algebra \mathcal{O} have very special properties due to their connections with projections in \mathcal{O}^{**} . These properties can be described geometrically in terms of the convex structure of K , and these geometric properties are used to define the general notion of a projective face of a convex set, which is a ("non-central") generalization of a split face. (See [4 ; p.12] for the definition; cf.

also [4; Th.3.8] for an equivalent characterization). By [4; Ths. 8.9 & 10.6] there exists a well behaved functional calculus (generalizing that of \mathcal{O}_{sa}) for the space $A(K)$ of any convex compact set K with "sufficiently many" projective faces. Here the term "sufficiently many" can be made precise in various (equivalent) ways; the most satisfactory seems to be that of [5; Th.2.2 & Prop.2.5] which is based on the concept of orthogonality (written $a \perp b$) for positive affine functions a, b . (See [4; p.44] for the definition of orthogonality, which is based on the notion of a projective face).

Note that if \mathcal{O} is a C^* -algebra, then the Jordan product in \mathcal{O}_{sa} can be expressed as follows:

$$(1.2) \quad a \circ b = \frac{1}{2}[(a+b)^2 - a^2 - b^2] ,$$

where the squares at the right hand side are given by the functional calculus. Since the functional calculus is now known to be determined by the geometry of K , this gives a rather explicit version of Kadison's result that the Jordan product is determined by the compact convex set K (or equivalently, by the order unit space $A(K)$).

The right hand side of (1.2) is meaningful as soon as $A(K)$ has functional calculus, but it will not define a Jordan product in general. (See [4; Figs. 8,10] for examples of low-dimensional compact convex sets which have sufficiently many projective faces, but are non-isomorphic to the state spaces of all JB-algebras of the appropriate dimensions).

A complete geometric characterization of the state spaces for JB-algebras was given in [6]. The main result of that paper states that a compact convex set K is affinely isomorphic to the

state space of a JB-algebra iff it has the following properties:

- (1.3) Every norm-exposed face of K is projective.
- (1.4) Every $A \in A(K)$ can be decomposed as $a = a_1 - a_2$ where $a_1, a_2 \in A(K)^+$ and $a_1 \perp a_2$.
- (1.5) The σ -convex hull of the extreme points of K is a split face.
- (1.6) The face generated by any two extreme points of K is norm-exposed and it is affinely isomorphic to the unit ball of some real Hilbert space.

The first two requirements above are related to spectral theory. The third one states that the state space is a direct convex sum of two faces, one being the σ -convex hull of the extreme points, the other containing no extreme points. This splitting into an "atomic" and a "non-atomic" part follows from well known facts in the case of a C^* -algebra. Finally, the fourth requirement is the Hilbert ball property which was mentioned in the beginning.

Passing to the case of a C^* -algebra, we can replace the Hilbert ball property by the much more restrictive 3-ball property, which has also been mentioned before. However, this strengthening of the axioms (1.3) - (1.6) will not suffice to yield the state space of a C^* -algebra. (A counterexample is given in § 6). In fact, the problem of characterizing state spaces of C^* -algebras is conceptually different from the similar problem for JB-algebras. Now, we have no explicit candidate like (1.2) for the product, and in fact there may exist different C^* -products on $A(K) + iA(K)$ determining the same Jordan product on $A(K)$, and hence the same state space K . Thus, the C^* -product has to be chosen, and the

missing axiom should be such as to make this choice possible.

The clue to this problem is the notion of "orientability", mentioned earlier. The first time a notion of orientation was used for a similar purpose, was in Connes' paper [14], where he gave a geometric characterization of the cones associated with von Neumann algebras via Tomita-Takesaki theory. Although both the setting and the actual definition are different in the two cases, they are related in spirit. In both cases the orientation serves the same purpose, namely to provide the complex Lie structure when the Jordan product is given. In Connes' paper, the Jordan structure of a von Neumann algebra is shown to be determined up to isomorphisms by the geometry of the associated cone $\mathcal{P}_{\xi_0}^{\#}$, which will be "autopolar", "facially homogeneous", and "orientable", and any chosen "orientation" provides a Lie product which together with the Jordan product will determine the von Neumann algebra. One suspects (but this remains open) that the first two properties mentioned above will suffice for a cone to yield a Jordan product. Some results in this direction have been achieved by Bellissard, Iochum and Lima in [10],[11],[12]. (In particular, it is shown in [11] that the conjecture is true if there exists a trace vector).

In the present paper, § 2 provides the necessary machinery of states and representations for JB-algebras. The results here are for the most part analogs of well known results for C^* -algebras.

In § 3 the 3-ball property is introduced and studied. It is shown that the state space of a C^* -algebra has the 3-ball property, and also that for a JB-algebra with the 3-ball property there is an irreducible representation on a Hilbert space associated with each pure state. However, unlike the situation for C^* -algebras,

this representation is not unique up to unitary equivalence; now we must also allow for conjugate linear isometries of the Hilbert space ("conjugations"). Thus, for each pure state the associated irreducible representation may "flip" from one unitary equivalence class to the conjugate one.

§ 4 provides a technical result which is also of some independent interest, namely that a JB-algebra with 3-ball property acts reversibly in each concrete representation as a Jordan operator algebra. This implies in particular that each such JB-algebra is isomorphic to the self-adjoint part of a "real C*-algebra", i.e. to the self-adjoint part of a norm closed real *-subalgebra of $B(H)$.

In § 5 we prove the key result that every JB-algebra A with the 3-ball property admits an "enveloping C*-algebra" \mathcal{O} with the universal property that every Jordan homomorphism $\varphi: A \rightarrow B(H)_{sa}$ can be extended to a *-homomorphism $\tilde{\varphi}: \mathcal{O} \rightarrow B(H)$ with the range of $\tilde{\varphi}$ being the C*-algebra generated by $\varphi(A)$, and that the enveloping C*-algebra is in a natural sense unique. Also it is shown that (except for possible "degeneracy" related to the existence of 1-dimensional representations) the restriction map is two-to-one from the pure states of \mathcal{O} onto those of A , and that there is a natural \mathbb{Z}_2 -action on the fibers, which is related to the "flip" alluded to above. Note that if A is a priori the self-adjoint part of a C*-algebra \mathcal{O}_0 then \mathcal{O} is in general different from (larger than \mathcal{O}_0).

In § 6 the notion of orientability is defined, and the relationship between consistent choice of orientations and consistent choice of irreducible representations associated with pure states, is explained.

§ 7 contains the main theorem which has already been stated.

In § 8 the orientability and related concepts are transferred from the pure states to the "spectrum" and the "primitive ideal space", and it is shown that K (which is supposed to have the 3-ball property) is orientable iff certain natural \mathbb{Z}_2 -bundles over the pure state space, the spectrum, and the primitive ideal space, are trivial.

Finally, § 9 contains a geometric characterization of the dual action of $*$ -homomorphisms between C^* -algebras. The key notion here is that of an "orientation preserving map", which provides the morphisms in the category of "oriented state spaces". (See Corollary 9.3 for the details).

§ 2. States and representations for JB-algebras.

This section is of preliminary nature, and the results are for the most part analogues of well known results for C^* -algebras.

Note that when we work in the context of Jordan algebras, we will use the word ideal to mean a norm closed Jordan ideal. Also if A, B are Jordan algebras and $T: A \rightarrow B$ is a bounded linear map, then we denote the adjoint map from B^* into A^* by T^* . Occasionally if $T: A^{**} \rightarrow B^{**}$ is a σ -weakly continuous linear map, we will denote the adjoint map from $B^* \rightarrow A^*$ by T^* . Recall also that a split face of a convex set K is a face F admitting a (necessarily unique) complementary face F' such that K is the direct convex sum of F and F' . (See [1; Ch.II, § 6] for further properties of split faces). The C^* -algebra version of the following proposition was established in [2].

Proposition 2.1. Let A be a JB-algebra with state space K . If J is an ideal of A , then the annihilator $J^\perp = J^0 \cap K$ of J in K is a w^* -closed split face. Conversely, if F is a w^* -closed split face of K , then the annihilator F_\circ of F in A is an ideal. Moreover, the mappings $J \mapsto J^\perp$ and $F \mapsto F_\circ$ are inverses.

Proof. Let J be an ideal in A . Then $J^{00} = J^- \subseteq A^{**}$ is a σ -weakly closed ideal in the enveloping JBW-algebra A^{**} , so $J^{00} = \text{im } U_c$ for some central idempotent $c \in A^{**}$ (cf [34; Lem.2.1]). Thus

$$(2.1) \quad J^\perp = J^0 \cap K = (\text{im } U_c)^0 \cap K = (\ker U_c^*) \cap K = (\text{im } U_{e-c}^*) \cap K.$$

Since $U_{e-c} + U_c = I$, it is easily verified that $(\text{im } U_{e-c}^*) \cap K$ is

a split face of K . (This also follows from the general result of [4; Prop.10.2]). Clearly, $J^\perp = J^0 \cap K$ is w^* -closed.

Now let F be any w^* -closed split face of K . By [4; Prop.10.2] and [5; Th.3.1] there exists a central idempotent $d \in A^{**}$ such that $F = (\text{im } U_d^*) \cap K$. Therefore, the annihilator $\ker U_d = \text{im } U_{e-d}$ of F in A^{**} is a Jordan ideal of A^{**} . Hence the intersection $F_0 = (\ker U_d) \cap A$ is a Jordan ideal in A . Clearly F_0 is norm closed.

To prove that $J \mapsto J^\perp$ and $F \mapsto F_0$ are inverse maps, we first observe that $(J^\perp)_0 = (J^0)_0 = J$, since $\text{im } U_{e-c}^*$ is positively generated and J^\perp is expressed by (2.1). Finally we will show $(F_0)^\perp = F$. Note that by [4; Prop.2.14] the unit ball of $\text{lin } F$ is $\text{co}(FU-F)$. Note also that this unit ball is w^* -compact since F is. By the Krein-Šmulian theorem $\text{lin } F$ is w^* -closed. Thus $(F_0)^0 = (\text{lin } F)_0^0 = \text{lin } F$, and so $(F_0)^\perp = (\text{lin } F) \cap K = F$. \square

Note that Proposition 2.1 corresponds to a C^* -algebra theorem relating (norm closed, 2-sided) ideals to w^* -closed split faces of the state space, cf. [2; § 7].

We next relate homomorphisms of JB-algebras to σ -weakly continuous homomorphisms of their enveloping JBW-algebras.

Lemma 2.2. If M_1 and M_2 are JBW-algebras and $\varphi: M_1 \rightarrow M_2$ is a σ -weakly continuous homomorphism, then $\varphi(M_1)$ is σ -weakly closed in M_2 , and so it is a JBW-algebra.

Proof. The unit ball of $\varphi(M_1)$ will be σ -weakly compact, and the result follows. (See [33; Prop.1.16.2] for the details of the analogous proof for von Neumann algebras). \square

Proposition 2.3. Let $\varphi : A \rightarrow M$ be a homomorphism from a JB-algebra A into a JBW-algebra M . Then there exists a unique σ -weakly continuous homomorphism $\tilde{\varphi} : A^{**} \rightarrow M$ which extends φ ; moreover $\tilde{\varphi}(A^{**}) = \varphi(A)^{-}$ (σ -weak closure).

Proof. The proof of the corresponding C^* -algebra result in [33; Prop.1.17.8 and 1.21.13] can be used without significant change. \square

Let K be the state space of a JB-algebra and suppose $\rho \in K$. Then by [5; Prop.1.12] there is a smallest split face F_ρ containing ρ . By [1; Prop.II.6.20] there is also a smallest w^* -closed split face containing ρ ; the next result shows that this face is just the w^* -closure of F_ρ . (Note that the corresponding result is incorrect for general compact convex sets, cf.e.g. [1; Ths.II.6.22 & II.7.19]. Note also that our notation here differs from that of [2] where F_ρ denotes the smallest w^* -closed split face containing ρ).

Proposition 2.4. If K is the state space of a JB-algebra A , then the w^* -closure \overline{F} of every split face F of K is again a split face.

Proof. Let $c \in A^{**}$ be the central idempotent such that $(\text{im } U_c) \cap K = F$, and let $J = (\ker U_c) \cap A$. Note that since c is central, $U_c : a \mapsto \{cac\} = c \circ a$ is a Jordan homomorphism, and so J is a Jordan ideal. From [7; Lem.9.3] Jordan isomorphisms are isometries, so $U_c : A/J \rightarrow A^{**}$ is isometric. Thus in the sense described in [17; § 6], c is regular, i.e. $\|c \circ a\| = \|a + J\|$ for $a \in A$. Now the proof of [17; Th.6.1] applies to complete the proof. For the convenience of the reader, we sketch the details.

By Proposition 2.1 there is a w^* -closed split face G of K such that $J^0 \cap K = G$ and $\text{lin } G = J^0$; the unit ball of $\text{lin } G$ will be $\text{co}(GU-G)$. Since $(A/J)^* \cong J^0$, we have for each $a \in A$:

$$\|a+J\| = \sup_{\substack{\rho \in J^0 \\ \|\rho\| \leq 1}} |\langle a, \rho \rangle| = \sup_{\rho \in \text{co}(GU-G)} \langle a, \rho \rangle.$$

On the other hand, since the unit ball of $\text{im } U_c^*$ is $\text{co}(FU-F)$, we also have for each $a \in A$:

$$\|c \circ a\| = \sup_{\rho \in K} |\langle a, U_c^* \rho \rangle| = \sup_{\rho \in F} |\langle a, \rho \rangle| = \sup_{\rho \in \text{co}(FU-F)} \langle a, \rho \rangle.$$

Since $\|c \circ a\| = \|a+J\|$, we can use a standard Hahn-Banach argument to show that $\text{co}(FU-F)$ is w^* -dense in $\text{co}(GU-G)$. Hence $\text{co}(\overline{FU}-\overline{F})$ is w^* -dense in $\text{co}(GU-G)$, and since these two sets are both w^* -compact, they are equal. Now suppose $\sigma \in G$. Then $\sigma = \lambda \rho + (1-\lambda)\rho'$ where $\rho \in \overline{F}$, $\rho' \in -\overline{F}$ and $0 \leq \lambda \leq 1$. Evaluating at e we get $\lambda = 1$, so $\sigma = \rho \in \overline{F}$, which completes the proof. \square

Definitions. A representation of a JB-algebra A is a homomorphism $\varphi: A \rightarrow M$ into a type I JBW-factor M . We say φ is a dense representation if $\varphi(A)^- = M$ (σ -weak closure). Two representations $\varphi_i: A \rightarrow M$ ($i=1,2$) are said to be Jordan equivalent if there exists an isomorphism Φ of M_1 onto M_2 such that $\varphi_2 = \Phi \circ \varphi_1$.

The purpose of the above definitions is to provide Jordan analogues of the basic notions in the representation theory of C^* -algebras. Since a JB-algebra might not have any (non-zero) representation into $B(H)_{sa}$, these notions can not be carried over directly. However, it seems reasonable to replace $B(H)$ by any JBW-factor of type I when we work with general JB-algebras. (Note that by [7; Th.8.6] and [32; Cor.2.4] a JBW-factor is

either isomorphic to M_3^8 or to one of the type I JW-factors. The latter have been classified in [36; Ths.5.2. & 7.1]; they are either spin factors or isomorphic to algebras which are essentially the bounded self-adjoint operators on real, complex, or quaternionic Hilbert space).

Observe that density and irreducibility are equivalent for a representation $\varphi: \mathcal{O} \rightarrow B(H)$ of a C^* -algebra [33; Prop.1.21.9]. Also, recall that two representations $\varphi_i: \mathcal{O} \rightarrow B(H)$ ($i=1,2$) of a C^* -algebra are unitarily equivalent iff there exists a $*$ -isomorphism Φ from $B(H_1)$ onto $B(H_2)$ such that $\varphi_2 = \Phi \circ \varphi_1$ [15; Cor.III.3.1]. Thus, dense representations and Jordan equivalence seem to be the appropriate Jordan analogues of irreducible representations and unitary equivalence for C^* -algebras.

Note, however, that the notion of Jordan equivalence will be less stringent than that of unitary equivalence when specialized to (the self-adjoint part of) C^* -algebras. In fact, if \mathcal{O} is a C^* -algebra and $\varphi_i: \mathcal{O}_{sa} \rightarrow B(H_i)_{sa}$ are Jordan equivalent representations, then the Jordan isomorphism $\Phi: B(H_1)_{sa} \rightarrow B(H_2)_{sa}$ connecting φ_1 and φ_2 will not necessarily be extendable to a $*$ -isomorphism of $B(H_1)$ onto $B(H_2)$. (e.g. consider the identity map and the transpose map on $M_2(\mathbb{C})_{sa}$).

We recall from [7] how one can associate with any given pure state ρ on a JB-algebra A a dense representation, namely $\varphi_\rho: A \rightarrow c(\rho) \circ A^{**}$ given by $\varphi_\rho(a) = c(\rho) \circ a$, where $c(\rho)$ is the central support of ρ , i.e. the smallest **central idempotent** $c \in A^{**}$ such that $\langle c, \rho \rangle = 1$. (See [7; Prop.5.6 and Prop.8.7] for the demonstration that φ_ρ is a dense representation).

Lemma 2.5. Let A be a JB-algebra with state space K , and let $\varphi_i : A \rightarrow M_i$ ($i = 1, 2$) be two dense representations. Then φ_1 and φ_2 are Jordan equivalent iff the unique σ -weakly continuous extensions $\tilde{\varphi}_i : A^{**} \rightarrow M_i$ satisfy $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$.

Proof. Suppose that φ_1 and φ_2 are equivalent, and let Φ be a Jordan isomorphism of M_1 onto M_2 such that $\varphi_2 = \Phi \circ \varphi_1$. Since Φ is σ -weakly continuous, we also have $\tilde{\varphi}_2 = \Phi \circ \tilde{\varphi}_1$, and so $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$.

Conversely, suppose $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$. By Lemma 2.2, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are surjective. Thus we can define $\Phi : M_1 \rightarrow M_2$ by $\Phi(\tilde{\varphi}_1(a)) = \tilde{\varphi}_2(a)$ for all $a \in A^{**}$. This Φ determines a Jordan equivalence of φ_1 and φ_2 . \square

The basic results on abstract factor representations of type I are summed up in the following proposition.

Proposition 2.6. Let A be a JB-algebra with state space K . If $\varphi : A \rightarrow M$ is a dense representation, then there exists $\rho \in \partial_e K$ such that φ is Jordan equivalent with φ_ρ ; now φ^* is an affine isomorphism of the normal state space of M onto the split face F_ρ generated by ρ , and F_ρ is the annihilator in K of $\ker \tilde{\varphi} \subseteq A^{**}$ (where $\tilde{\varphi} : A^{**} \rightarrow M$ is the σ -weakly continuous extension of φ as before), while the w^* -closed split face \overline{F}_ρ generated by ρ is the annihilator in K of $\ker \varphi \subseteq A$. If $\rho, \sigma \in \partial_e K$, then it is a necessary and sufficient condition for Jordan equivalence of φ_ρ and φ_σ that ρ and σ are not separated by a split face, i.e. that $\overline{F}_\rho = \overline{F}_\sigma$.

Proof: Since $\ker \tilde{\varphi}$ is a σ -weakly closed ideal in A^{**} , there exists a central idempotent $c \in A^{**}$ such that $\ker \tilde{\varphi} = (e-c) \cdot A^{**}$. [34; Lem.2.1]. Observe that the restriction of $\tilde{\varphi}$

to $c \circ A^{**}$ is an isomorphism onto M . (Surjectivity follows by Lemma 2.2). The predual of $c \circ A^{**}$ is $\text{im } U_c^*$ and the normal state space is $F = (\text{im } U_c^*) \cap K$ (cf. [5; Prop.1.10]). Since $c \circ A^{**}$ is a factor, then F will contain no proper split faces. Since $c \circ A^{**}$ is of type I, then F will contain at least one extreme point $\rho \in \partial_e K$. Thus $F = F_\rho$. Now $c = c(\rho)$, so $\ker \tilde{\varphi}_\rho = (e-c) \circ A^{**} = \ker \tilde{\varphi}$. By Lemma 2.5, φ and φ_ρ must be Jordan equivalent.

Since (the restriction of) $\tilde{\varphi}$ is a Jordan isomorphism of $c \circ A^{**}$ onto M , then φ^* must be an affine isomorphism of the normal state space of M onto the normal state space $F = F_\rho$ of $c \circ A^{**}$.

Working in the duality of A^* and A^{**} , we have

$$F_\rho = (\text{im } U_c^*) \cap K = (\ker U_c)^\perp = [(e-c) \circ A^{**}]^\perp.$$

Hence F_ρ is the annihilator in K of $\ker \tilde{\varphi} \in A^{**}$.

Dually we have

$$F_\rho^\circ = \ker U_c = (e-c) \circ A^{**} = \ker \tilde{\varphi}.$$

Thus the annihilator of F_ρ in A will be $\ker \varphi = (\ker \tilde{\varphi}) \cap A$. Clearly \overline{F}_ρ has the same annihilator in A as F_ρ , so $\ker \varphi$ corresponds to the w^* -closed split face \overline{F}_ρ (cf. Proposition 2.4) under the correspondence established in Proposition 2.1. Hence \overline{F}_ρ is the annihilator in K of $\ker \varphi \subseteq A$.

Finally, $\ker \tilde{\varphi}_\rho = \ker \tilde{\varphi}_\sigma$ if $c(\rho) = c(\sigma)$. This completes the proof. \square

We are now in a position to transfer to JB-algebras the well known definition and basic properties of the Jacobson hull-kernel topology on the primitive ideal space of a C^* -algebra

(cf. [23] and also [16; § 3.1]). This can be done by specializing the theory of the hull-kernel topology for M -ideals in a Banach space, since the ideals of a JB -algebra A are precisely the M -ideals of A considered as a Banach space. (This follows by Proposition 2.1 and [3; Cor.5.9.]). However, we prefer to give a direct presentation, which is almost equally short.

By definition, a primitive ideal of a C^* -algebra is the kernel of an irreducible representation. In view of our earlier remarks, we are led to the following definition for a JB -algebra:

Definitions. An ideal J in a JB -algebra A is primitive if it is the kernel of a dense representation. A split face of the state space K of A is primitive if it is the annihilator of some primitive ideal in A , or what is equivalent (by Proposition 2.6), if it is the smallest w^* -closed split face \overline{F}_ρ containing a given extreme point $\rho \in \partial_e K$. The hull of an ideal J in A is the collection $h(J)$ of all primitive ideals containing J .

Proposition 2.7. The set of primitive ideals of a JB -algebra A can be equipped with a compact T_0 -topology whose closed sets are the hulls $h(J)$ where J is any ideal of A . Pulling this topology back by the map $\rho \mapsto (\overline{F}_\rho)^0$, we obtain the facial topology of $\partial_e K$, whose closed sets are the intersections of $\partial_e K$ by w^* -closed split faces.

Proof. As in the corresponding proof for C^* -algebras [16; § 3.1], we need the key result that every primitive ideal J in A is prime, i.e. if J_1 and J_2 are two ideals such that $J_1 \cap J_2 \subseteq J$ then $J_1 \subseteq J$ or $J_2 \subseteq J$.

By Proposition 2.1, $(J_1 \cap J_2)^\perp$ is the smallest w^* -closed

split face containing the split faces J_1^\perp and J_2^\perp . By [1; Prop.II.6.8] the convex hull of two split faces is a split face, so $(J_1 \cap J_2)^\perp = \text{co}(J_1^\perp \cup J_2^\perp)$. Now by Proposition 2.6, $J^\perp = \overline{F}_\rho$ for some $\rho \in \partial_e K$, so the assumed inclusion $J_1 \cap J_2 \subseteq J$ implies

$$\overline{F}_\rho = J^\perp \subseteq (J_1 \cap J_2)^\perp = \text{co}(J_1^\perp \cup J_2^\perp).$$

Hence $\rho \in J_1^\perp$ or $\rho \in J_2^\perp$, and thus in turn $\overline{F}_\rho \subseteq J_1^\perp$ or $\overline{F}_\rho \subseteq J_2^\perp$. This means that $J_1 \subseteq J$ or $J_2 \subseteq J$, so we have shown that J is prime.

Now it is straightforward to show that the set of primitive ideals is a T_0 -space for a topology whose closed sets are the hulls. Also it is easily seen that the topology of $\partial_e K$ obtained by pulling back this topology by the map $\rho \mapsto (F_\rho)^\circ$ is precisely the facial topology defined in the proposition. (This topology was defined for arbitrary compact convex sets in [2; § 4]). In general, the facial topology of the extreme boundary of a compact convex set is compact (but possibly non-Hausdorff) by [1; Prop.II.6.21]; hence the described topology on the collection of primitive ideals of A is compact. \square

In the sequel we will denote the collection of all primitive ideals of a JB-algebra A by the symbol $\text{Prim}(A)$, and we will assume that it is equipped with the topology described above, which we will call the structure topology (or the "hull-kernel" topology).

We now turn to a notion of spectrum for JB-algebras, which will generalize the spectrum of C^* -algebras. It follows from our previous remarks that the Jordan equivalence classes of dense representations of a JB-algebra are the analogues of the unitary equivalence classes of irreducible representations of a C^* -algebra.

This leads us to define the spectrum \hat{A} of a JB-algebra A to be the Jordan equivalence classes of dense representations.

We next define the topology on the spectrum. If φ is a dense representation of a JB-algebra A , then we will denote the class of all dense representations which are Jordan equivalent with φ , by the symbol $[\varphi]$. (When no confusion will result, we will omit the brackets). It is clear that if φ_1 and φ_2 are Jordan equivalent, then $\ker \varphi_1 = \ker \varphi_2$. (By Proposition 2.6 we even have $\ker \tilde{\varphi}_1 = \ker \tilde{\varphi}_2$, and this equality characterizes Jordan equivalence). Thus there is a well defined map $[\varphi] \mapsto \ker \varphi$ from the Jordan equivalence classes of dense representations onto the primitive ideals. In the sequel we will assume that the spectrum \hat{A} of a JB-algebra is equipped with the topology obtained by pulling back the structure topology of $\text{Prim} A$ by the mapping $[\varphi] \mapsto \ker \varphi$.

By Proposition 1.6, the mapping $F_\rho \mapsto [\varphi_\rho]$ will map the collection of all split faces of the form F_ρ with $\rho \in \partial_e K$, bijectively onto \hat{A} . For convenience we write $\hat{K} = \{F_\rho \mid \rho \in \partial_e K\}$, and we will call the set \hat{K} equipped with the topology transferred from \hat{A} , the spectrum of the state space K of the given JB-algebra A . Similarly we write $\text{Prim} K = \{\bar{F}_\rho \mid \rho \in \partial_e K\}$, and we will call $\text{Prim} K$ equipped with the topology transferred from $\text{Prim} A$, the structure space of K . Clearly the transition from \hat{A} and $\text{Prim} A$ to \hat{K} and $\text{Prim} K$ is non-essential; it is merely a matter of convenience in view of our geometric approach. (We remark that unlike the situation with the primitive split faces \bar{F}_ρ , each F_ρ is actually minimal among all split faces and not only among those containing ρ . The difference stems from the fact that the complement of a split face is always a split face, while the complement of a w^* -closed split face is not a w^* -closed split face in general).

By the above definitions we have two natural surjections $\rho \mapsto [\varphi_\rho] \mapsto \ker \varphi_\rho$ of $\partial_e K \rightarrow \hat{A} \rightarrow \text{Prim} A$. If $\partial_e K$ is equipped with the facial topology and the other two spaces are topologized as explained above, then all these maps are continuous and open. Note that the **canonical** map from $\partial_e K$ with w^* -topology onto $\text{Prim} K$ with facial topology is continuous. In the C^* -algebra case, this map is also open. However, this is a non-trivial result involving the Kadison transitivity theorem [16; Th.3.4.11], and it is an open question if the corresponding result holds in our general setting.

Clearly the above surjections can be transferred from \hat{A} and $\text{Prim} A$ to \hat{K} and $\text{Prim} K$. Since these "geometric" counterparts will be used repeatedly in the sequel, we find it convenient to restate the above results in terms of these maps.

Proposition 2.8. If K is the state space of a JB-algebra and $\partial_e K$ is equipped with the facial topology, then the maps $\partial_e K \rightarrow \hat{K} \rightarrow \text{Prim} K$ defined by $\rho \mapsto F_\rho \mapsto \overline{F}_\rho$ are continuous and open surjections. In particular, a subset of \hat{K} (or of $\text{Prim} K$) is closed iff it is the collection of all F_ρ (respectively \overline{F}_ρ) contained in a fixed closed split face of K .

Proof. Evident. \square

We will now investigate unital homomorphisms of JB-algebras and the dual maps between their state spaces.

Lemma 2.9. If $\varphi: A_1 \rightarrow A_2$ is a unital homomorphism between two JB-algebras A_1 and A_2 with state spaces K_1 and K_2 , then for each ideal J in A_2 we have $\varphi^*(J^\perp) = \varphi^{-1}(J)^\perp$.

Proof. 1.) We first consider $J = \{0\}$. Then $J^\perp = K_2$ and $\varphi^{-1}(J) = \ker \varphi$, so we must prove $\varphi^*(K_2) = (\ker \varphi)^\perp$.

If $\rho \in K_2$ and $a \in \ker \varphi$, then $\langle a, \varphi^*(\rho) \rangle = \langle \varphi(a), \rho \rangle = 0$; so we have shown $\varphi^*(K_2) \subseteq (\ker \varphi)^\perp$.

If $\sigma \in (\ker \varphi)^\perp$, then we define ρ_1 by $\langle \varphi(a), \rho_1 \rangle = \langle a, \sigma \rangle$ for $a \in A_1$. Now, ρ_1 is a state on $\varphi(A_1)$, so it can be extended to a state ρ on A_2 . By definition $\langle a, \varphi^*(\rho) \rangle = \langle a, \sigma \rangle$ for all $a \in A_1$, so $\varphi^*(\rho) = \sigma$. Thus $\varphi^*(K_2) = (\ker \varphi)^\perp$.

2.) Assume next that J is an arbitrary ideal in A_2 , and let $\psi : A_2 \rightarrow A_2/J$ be the quotient map and K'_2 the state space of A_2/J . Then by the result in the first part of the proof

$$\begin{aligned} \varphi^*(J^\perp) &= \varphi^*((\ker \psi)^\perp) = \varphi^*(\psi^*(K'_2)) \\ &= (\psi \circ \varphi)^*(K'_2) = (\ker \psi \circ \varphi)^\perp = \varphi^{-1}(J)^\perp. \quad \square \end{aligned}$$

Remark. The same result as in Lemma 2.9 will hold, with the same proof, for a σ -weakly continuous unital homomorphism between JBW-algebras and for σ -weakly continuous ideals. Specifically, let A_1 and A_2 be JBW-algebras with normal state spaces K_1 and K_2 , and let $\Phi : A_1 \rightarrow A_2$ be a σ -weakly continuous homomorphism defining a (predual) map $\Phi^* : K_2 \rightarrow K_1$; then for each σ -weakly closed ideal J in A_2 we have $\Phi^*(J^\perp) = \Phi^{-1}(J)^\perp$.

Proposition 2.10. If $\varphi : A_1 \rightarrow A_2$ is a unital homomorphism between two JB-algebras A_1 and A_2 , then the dual map $\varphi^* : K_2 \rightarrow K_1$ between their state spaces will take split faces onto split faces and w^* -closed split faces onto w^* -closed split faces.

Proof. To prove the first statement, we work in the spectral duality of the enveloping algebra A_j^{**} and its predual A_j^* for

$j = 1, 2$. Now the map $F \mapsto J = F^\circ$ is known to be a bijection of all split faces of K_j onto all σ -weakly closed ideals of A_j^{**} , the inverse map being $J \mapsto F = J^\perp$ (cf. proof of Proposition 2.1). Thus, for a given split face F of K_2 we consider the annihilator ideal $J = F^\circ$ in A_2^{**} , and then we apply the above "Remark" with $\Phi = \varphi^{**} : A_1^{**} \rightarrow A_2^{**}$ (and $\Phi_* = \varphi^*$). This gives $\varphi^*(F) = (\varphi^{**})^{-1}(J)_\perp$, where $(\varphi^{**})^{-1}(J)$ is a σ -weakly closed ideal in A_1^{**} . Hence $\varphi^*(F)$ is a split face of K_2 , as desired.

Finally we note that the w^* -continuous map φ^* will map w^* -continuous sets into w^* -continuous sets; from this the last statement of the proposition follows. \square

We will close this section by studying the one-dimensional representations of JB-algebras. Note first that if A is a JB-algebra with state space K and if $\rho \in \partial_e K$, then $\dim \varphi_\rho(A) = 1$ iff $F_\rho = \{\rho\}$, i.e. iff $\{\rho\}$ is a split face of K (cf. Proposition 2.6). Thus $\dim \varphi_\rho(A) = 1$ iff $\dim F_\rho = 0$.

For convenience we introduce the following notation:

$$(2.2) \quad \partial_{e,0} K = \{\rho \in \partial_e K \mid F_\rho = \{\rho\}\},$$

$$(2.3) \quad (\hat{K})_0 = \{F_\rho \mid F_\rho = \{\rho\}\},$$

$$(2.4) \quad \text{Prim}_0 K = \{\overline{F} \mid F_\rho = \{\rho\}\}.$$

Also we write $\partial_{e,1} K = \partial_e K \setminus \partial_{e,0} K$, $(\hat{K})_1 = \hat{K} \setminus (\hat{K})_0$ and $\text{Prim}_1 K = \text{Prim} K \setminus \text{Prim}_0 K$.

Proposition 2.11. Let A be a JB-algebra with state space K . Let

$$J = \bigcap_{\rho \in \partial_{e,0} K} \ker \varphi_\rho,$$

and let $F = J^\perp$. (If $\partial_{e,0} K = \emptyset$, set $J = A$ and $F = \emptyset$).

Then A/J is associative and $\partial_e F = \partial_{e,0} K$.

In particular, $\partial_{e,o} K$ is facially closed in $\partial_e K$, $(\hat{K})_o$ is closed in \hat{K} , and $\text{Prim}_o K$ is closed in $\text{Prim } K$.

Proof. For each $\rho \in \partial_{e,o} K$, $\varphi_\rho(A)$ is associative, so $\ker \varphi_\rho$ contains $a \circ (b \circ c) - (a \circ b) \circ c$ for any given triple $a, b, c \in A$. It follows that J also contains all such expressions. Hence A/J is associative.

Now suppose $\rho \in \partial_e F = F \cap \partial_e K$. Then $F_\rho \subseteq F$, so by dualization (and use of Propositions 2.1 & 2.6):

$$J \subseteq (F_\rho)_o = (\overline{F}_\rho)_o = ((\ker \varphi_\rho)^\perp)_o = \ker \varphi_\rho.$$

Hence, φ_ρ factors through A/J , so $\varphi_\rho(A)^\perp$ will be associative, and thus by [7; Prop.2.3] it will be isomorphic to $C(X)$; since it is a JBW-factor it must be one-dimensional. Thus $\rho \in \partial_{e,o} K$. Then (by Proposition 2.6)

$$\rho \in \overline{F}_\rho = (\ker \varphi_\rho)^\perp \subseteq J^\perp = F,$$

so $\rho \in F \cap \partial_e K = \partial_e F$. Thus we have proved $\partial_e F = \partial_{e,o} K$.

The last statement of the proposition follows from the statement just proved by virtue of the definition of the topologies involved. \square

We close this section by giving a geometric characterization of the (dual version of) unital Jordan homomorphisms. Recall that by definition a face F of K is norm exposed if $F = a^{-1}(0)$ for some positive affine function a on K , or what is equivalent, for $a \in (A^{**})^+$. In [4; § 12] it was shown that for each such F there is a unique idempotent $p = r(a) \in A^{**}$ such that $F = p^{-1}(0)$. Then $p^{-1}(1) = (e-p)^{-1}(0)$ is also norm exposed, and is denoted $F^\#$; F and $F^\#$ are said to be quasicomplementary projective faces. Finally, we recall that $a, b \in A^+$ are orthogonal if there exists a

norm exposed face F with $a = 0$ on F and $b = 0$ on $F^\#$. (Note that for state spaces of JB-algebras this definition will coincide with that of [4], since every norm exposed face is projective).

We can now state the characterization.

Proposition 2.12. Let $\psi: K_2 \rightarrow K_1$ be a w^* -continuous affine map between state spaces K_2 and K_1 of JB-algebras A_2 and A_1 . Then ψ is the dual of a unital Jordan homomorphism from A_1 into A_2 iff ψ^{-1} preserves quasicomplements, i.e. $\psi^{-1}(F^\#) = \psi^{-1}(F)^\#$ for every projective face F of K_1 .

Proof. Assume first that $\varphi: A_1 \rightarrow A_2$ is a unital Jordan homomorphism such that $\varphi^* = \psi$, and let F be a projective face in K_1 , say $F = p^{-1}(0)$ for $p^2 = p \in A_1^{**}$. Then

$$\psi^{-1}(F) = \psi^{-1}(p^{-1}(0)) = (\varphi^{**}(p))^{-1}(0),$$

while

$$\psi^{-1}(F^\#) = \psi^{-1}(p^{-1}(1)) = (\varphi^{**}(p))^{-1}(1).$$

Since $\varphi^{**}: A_1^{**} \rightarrow A_2^{**}$ is a Jordan homomorphism, then $\varphi^{**}(p)$ is an idempotent, so we have shown that ψ^{-1} preserves quasicomplements.

Conversely, suppose ψ^{-1} preserves quasicomplements. We first show that ψ^{-1} sends projective faces to projective faces. If $p^2 = p \in A_1^{**}$ and $F = p^{-1}(0)$, then

$$\psi^{-1}(F) = \psi^{-1}(p^{-1}(0)) = (p \circ \psi)^{-1}(0).$$

Since $p \circ \psi \in (A_2^{**})^+$, then $\psi^{-1}(F)$ is a norm exposed, and hence projective, face of K_2 .

Next we show that ψ preserves orthogonality of elements of A_1^+ . Suppose $a, b \in A_1^+$ and $a \perp b$. Let F be a norm exposed, hence projective, face of K_1 such that $a = 0$ on F and $b = 0$ on $F^\#$.

Now $\varphi(a)$ and $\varphi(b)$ are positive elements of A_2 which are zero on $\psi^{-1}(F)$ and $\psi^{-1}(F^\#) = \psi^{-1}(F)^\#$ respectively, and so $\varphi(a) \perp \varphi(b)$.

Now suppose a is any element of A_1 , with orthogonal decomposition $a = a^+ - a^-$. By virtue of the uniqueness of the orthogonal decomposition (cf. [5]) we conclude that $\varphi(a^+) - \varphi(a^-)$ is the orthogonal decomposition of $\varphi(a)$ in A_2 ; in particular $\varphi(a^+) = \varphi(a)^+$.

Since φ is positive and unital, then $\|\varphi\| \leq 1$. Now the set of all $f \in C(\sigma(a))$ such that $\varphi(f(a)) = f(\varphi(a))$ is seen to be a norm closed vector sublattice of $C(\sigma(a))$; by the Stone-Weierstrass theorem it equals $C(\sigma(a))$. In particular φ will preserve squares and then also Jordan products. Thus φ is a Jordan homomorphism. \square

§ 3. The 3-ball property.

It was shown in [6; Cor.3.12] that if K is the state space of a JB-algebra, then the face $B(\rho, \sigma)$ generated by an arbitrary pair of extreme points ρ, σ is affinely isomorphic to a Hilbert ball (i.e. the closed unit ball of some real Hilbert space). In general all possible dimensions for the balls $B(\rho, \sigma)$ can occur (cf. [6; Lem.3.10]). However, if K is the state space of a C^* -algebra, then the only possible dimensions are one and three, with the former representing a kind of degeneracy. This result is actually implicit in the argument leading up to [6; Th.3.11], but for the sake of completeness we will give the proof (Proposition 3.3). Thus we are led to the following general notion:

Definition. A convex set K has the 3-ball property if the face $B(\rho, \sigma)$ is a Hilbert ball of dimension one or three for each pair of distinct point $\rho, \sigma \in \partial_e K$. For brevity we shall also say that a JB-algebra has the 3-ball property if its state space has this property.

We will work with the 3-ball property mainly for state spaces of JB-algebras, but occasionally also for the normal state spaces of JBW-algebras. Note that the latter are more general than the former since the state space of any JB-algebra can be identified with the normal state space of its enveloping JBW-algebra (cf. [33]).

It follows from [6; Prop.3.1] that if K is any convex set and $\rho, \sigma \in \partial_e K$ are separated by a split face F (i.e. $\rho \in F$ and $\sigma \in F'$), then $B(\rho, \sigma)$ is just the line segment $[\rho, \sigma]$, i.e. a one-dimensional Hilbert ball. If K is the normal state space of a JBW-algebra, then the converse also holds:

Lemma 3.1. Let ρ, σ be extreme points of the normal state space of a JBW-algebra. Then $\dim B(\rho, \sigma) = 1$ iff ρ and σ are separated by a split face.

Proof. Suppose that ρ and σ are not separated by any split face. Then it follows from the proof of [6 ; Th.3.11] that $B(\rho, \sigma)$ is the normal state space of a spin factor. Every spin factor is of dimension at least three, so the affine dimension of $B(\rho, \sigma)$ will be at least two. \square

Corollary 3.2. The normal state space of a JBW-algebra has the 3-ball property iff $\dim B(\rho, \sigma) = 3$ for every pair of distinct extreme points ρ, σ not separated by a split face.

Proof. The normal state space of a JBW-algebra has the Hilbert ball property by [; Th.3.11]. \square

Proposition 3.3. The normal state space of any von Neumann algebra (and in particular the state space of any C^* -algebra) has the 3-ball property).

Proof. Let K be the normal state space of a von Neumann algebra \mathcal{M} , and let ρ, σ be distinct extreme points of K not separated by any split face. We denote by p and q the support projections of ρ and σ , i.e. the minimal projections in \mathcal{M} such that $\langle p, \rho \rangle = \langle q, \sigma \rangle = 1$. As shown in [6 ; proof of Th.3.11] the face $B(\rho, \sigma)$ of K is affinely isomorphic to the normal state space of $(p \vee q)\mathcal{M}(p \vee q)$. Note that $u, v \leq c(\rho)$ where $c(\rho) (= c(\sigma))$ is the central support projection of ρ . (The equality of $c(\rho)$ and $c(\sigma)$ follows since the central projections of \mathcal{M} are in 1-1 correspondence with the split faces

of K , and ρ and σ are supposed not to be separated by any split face). Hence $(p \vee q)\mathcal{O}(p \vee q) = (p \vee q)\mathcal{O}_\rho(p \vee q)$ where $\mathcal{O}_\rho = c(\rho)\mathcal{O}$. Furthermore, \mathcal{O}_ρ is a type I von Neumann factor (cf. [6; Lem.7.1]), and so $\mathcal{O}_\rho \cong B(H)$ for some Hilbert space H . The minimal projections $p, q \in \mathcal{O}_\rho$ will correspond to projections of rank one in $B(H)$, and so $p \vee q$ will correspond to a projection of rank two. Thus

$$(3.1) \quad (p \vee q)\mathcal{O}_\rho(p \vee q) \cong M_2(\mathbb{C}),$$

and so $B(\rho, \sigma)$ will be affinely isomorphic to the state space (= normal state space) of $M_2(\mathbb{C})$. But the state space of $M_2(\mathbb{C})$ is known to be a three-dimensional Euclidean ball (see e.g. [4; end of § 11]). \square

Remark. The proof of [6; Th.3.11] which was quoted above, is given in the context of JBW-algebras. But the only results on JBW-algebras which are used in the relevant part of the proof, are those which generalize well known results on von Neuman algebras (in particular the relationship between faces of K and projections in \mathcal{O} , cf. [17], [31]). Thus, if one wishes, one can give an alternative (if somewhat longer) proof of the von Neumann algebra result stated in Proposition 3.3 by means of conventional notions from von Neumann algebra theory.

Corollary 3.4. If ρ, σ are pure states on a C^* -algebra \mathcal{O} and π_ρ, π_σ are the corresponding GNS-representations, then $\dim B(\rho, \sigma) = 3$ when π_ρ and π_σ are unitarily equivalent and $\dim B(\rho, \sigma) = 1$ otherwise.

Proof. Specializing the proof of Proposition 2.6 to the C^* -algebra context, we find that π_ρ and π_σ are unitarily equivalent iff ρ and σ are not separated by a split face; by the proof of Proposition 3.3, $\dim B(\rho, \sigma) = 3$ in this case and $\dim B(\rho, \sigma) = 1$ otherwise. \square

The next proposition is crucial for the study of JB-algebra state spaces with the 3-ball property.

Proposition 3.5. A JBW-factor of type I is isomorphic to $B(H)_{sa}$ (the bounded self-adjoint operators on some Hilbert space H) iff its normal state space has the 3-ball property.

Proof. Let A be a JBW-factor of type I with normal state space J . If $A \cong B(H)_{sa}$, then the 3-ball property for K follows by Proposition 3.3. Conversely we assume that K has the 3-ball property, and we will show $A \cong B(H)_{sa}$. Recall in this connection that the correspondence between idempotents in A and projective faces of K will associate the central idempotents of A with the split faces of K (cf. [5; Prop.3.1] and [4; Prop.10.2]). Since A is a factor, there is no proper split face of K . Thus it follows from Lemma 3.1 that $\dim B(\rho, \sigma) = 3$ for every pair ρ, σ of distinct extreme points.

Assume first that A is of type I_2 . Then A is a spin factor [7; Prop.7.1], so K is a Hilbert ball [6; Lem.3.10]. Now $B(\rho, \sigma) = K$ for any pair $\rho, \sigma \in \partial_e K$, $\rho \neq \sigma$. Hence K must be a three-dimensional ball. But the only spin factor with such a normal state space is the algebra $M_2(\mathbb{C})_{sa}$ of dimension four (over \mathbb{R}). Thus, in this case $A \cong B(H)_{sa}$ with $\dim H = 2$.

Assume henceforth that A is of type I_n with $n \geq 3$. By [7; Prop.8.6] A is isomorphic to a JC-algebra, i.e. a norm

closed Jordan subalgebra of $B(H)_{sa}$ for some Hilbert space H . Since A is a JBW-algebra, it follows from [34; Cor.2.4] that A is isomorphic to a JW-algebra, i.e. a weakly closed Jordan subalgebra of $B(H)_{sa}$. Finally, by [36; Th.5.1] A can be faithfully represented as an irreducible JW-algebra. For the rest of the proof we assume that A is so represented, $A \subseteq B(H)_{sa}$.

By [7; Th.6.10] we can find a family $\{p_\alpha\}$ of orthogonal minimal idempotents in A such that $\sum_\alpha p_\alpha = 1$. By the correspondence between idempotents in A and projective faces of K , it follows that for $\alpha \neq \beta$ the normal state space of

$$A_{\alpha\beta} = (p_\alpha + p_\beta)A(p_\alpha + p_\beta)$$

can be identified with $B(\rho_\alpha, \rho_\beta)$ where $\{\rho_\alpha\}$ and $\{\rho_\beta\}$ correspond to p_α and p_β respectively. (Cf. the proof of [6; Th.3.11]). By hypothesis $\dim B(\rho_\alpha, \rho_\beta) = 3$, so $\dim A_{\alpha\beta} = 4$ for each pair α, β with $\alpha \neq \beta$.

For each pair α, β with $\alpha \neq \beta$ we consider the map $x \mapsto x + x^*$ of $p_\alpha A p_\beta \subseteq B(H)$ into $B(H)_{sa}$. This map is seen to be a real linear isomorphism onto the subspace

$$\{p_\alpha A p_\beta\} = \{p_\beta a p_\alpha + p_\beta a p_\alpha \mid a \in A\}.$$

Since

$$A_{\alpha\beta} = (p_\alpha A p_\alpha) \oplus (p_\beta A p_\beta) \oplus \{p_\alpha A p_\beta\},$$

and since $p_\alpha A p_\alpha$ and $p_\beta A p_\beta$ are both of dimension one, it follows that $\{p_\alpha A p_\beta\}$ is of dimension two (over \mathbb{R}).

In [36] the irreducible type I_n factors with $n \geq 3$ are classified by the dimension of the spaces $p_\alpha A p_\beta$. Since we have shown them to be two-dimensional (over \mathbb{R}), we must have the middle (i.e. the "complex") alternative of [36; Th.3.9]. Thus

for $\alpha \neq \beta$ we have $p_\alpha A p_\beta = \mathbb{C} w_{\alpha\beta}$ where $w_{\alpha\beta}$ is a partial isometry with initial projection p_α and final projection p_β , and where the ranges of p_α and p_β are one-dimensional subspaces of the (complex) Hilbert space H .

It follows that for $\alpha \neq \beta$ the real linear space

$$(p_\alpha + p_\beta)B(H)_{sa}(p_\alpha + p_\beta)$$

is of dimension four. Since it contains $A_{\alpha\beta}$ and the latter is known to be of dimension four, the two must be equal. From this it follows that

$$(3.2) \quad p_\alpha A p_\beta = p_\alpha B(H)_{sa} p_\beta, \quad \text{for } \alpha \neq \beta.$$

Thus, if $\alpha_1, \dots, \alpha_m$ are distinct indices, then

$$(\sum_i p_{\alpha_i}) A (\sum_j p_{\alpha_j}) = \sum_{i,j} p_{\alpha_i} A p_{\alpha_j} = (\sum_i p_{\alpha_i}) B(H)_{sa} (\sum_j p_{\alpha_j}).$$

Hence $(p_{\alpha_1} + \dots + p_{\alpha_m}) b (p_{\alpha_1} + \dots + p_{\alpha_m}) \in A$ for every $b \in B(H)_{sa}$. Since A is strongly closed, since $\sum_\alpha p_\alpha = 1$ (strong convergence), and since multiplication is strongly continuous on the unit ball of $B(H)$, it follows that $b \in A$ for every $b \in B(H)_{sa}$. Thus $A = B(H)_{sa}$ as desired. \square

Passing to a JB-algebra A with state space K , we shall see that the 3-ball property for K will hold iff the conclusion of Proposition 2.4 holds "locally" at each pure state $\rho \in \partial_e K$. As in [6; § 7] we use the notation $A_\rho = c(\rho)A^{**}$ where $c(\rho)$ is the central support of ρ .

Corollary 3.6. The state space K of a JB-algebra A has the 3-ball property iff $A_\rho \cong B(H)_{sa}$ for all $\rho \in \partial_e K$.

Proof. By Corollary 3.2 K has the 3-ball property iff F_ρ has the 3-ball property for each $\rho \in \partial_e K$. Now the corollary follows from Proposition 3.5 since each F_ρ can be identified with the normal state space of the corresponding type I JBW-factor A_ρ (cf. Proposition 2.6). \square

It is sometimes convenient to use the term "concrete representation" of a JB-algebra A to denote a homomorphism $\pi: A \rightarrow B(H)_{sa}$ where H is some Hilbert space. Note that for concrete representations the customary notion of irreducibility makes sense. Specifically, an irreducible representation of a JB-algebra A is a concrete representation $\pi: A \rightarrow B(H)_{sa}$ such that no proper (closed) subspace of H is invariant under $\pi(A)$.

We will now show that for JB-algebras with the 3-ball property, the notions of dense and irreducible representations are essentially the same.

Proposition 3.7. Let A be a JB-algebra with the 3-ball property. If $\varphi: A \rightarrow M$ is a dense representation, then $M \cong B(H)_{sa}$ for a suitable Hilbert space H . If $\pi: A \rightarrow B(H)_{sa}$ is any (concrete) representation, then π is dense iff it is irreducible.

Proof. By Proposition 2.6, φ is Jordan equivalent to $\varphi_\rho: A \rightarrow A_\rho$ for some $\rho \in \partial_e K$. By Corollary 3.6, $A_\rho \cong B(H)_{sa}$, and so by the definition of Jordan equivalence, $M \cong B(H)_{sa}$.

Now suppose that $\pi: A \rightarrow B(H)_{sa}$ is irreducible. Then $\pi(A)^-$ is an irreducible JW-algebra. (As before, the bar denotes weak closure). Irreducibility of a JW-algebra implies that it is a factor, and by [37; Th.4.1] irreducible JW-factors are of type I. Now the proof of Proposition 3.5 shows that $\pi(A)^- = B(H)_{sa}$, i.e. π is a dense representation.

Conversely, suppose $\pi: A \rightarrow B(H)_{sa}$ is dense, and suppose for contradiction that it is not irreducible. Then there exists a non-trivial projection $P \in B(H)$ such that P commutes with everything in $\pi(A)$. The commutant of $\{P\}$ is

$$PB(H)P + (I-P)B(H)(I-P),$$

so $\pi(A)$ is contained in

$$PB(H)_{sa}P + (I-P)B(H)_{sa}(I-P);$$

this contradicts the density of $\pi(A)$ in $B(H)_{sa}$. \square

Definition. Let A be a JB-algebra with the 3-ball property. We say an irreducible representation $\pi: A \rightarrow B(H)_{sa}$ is associated with $\rho \in \partial_e K$ if there exists a (unit) vector $\xi \in H$ such that

$$(3.3) \quad \langle a, \rho \rangle = (\pi(a)\xi | \xi) \quad \text{for all } a \in A.$$

Note that the unit vector ξ of (3.3) is uniquely determined up to scalar multiples (of modulus one) by virtue of the density of $\pi(A)$ in $B(H)_{sa}$. We will say that this vector ξ represents ρ w.r. to π .

Proposition 3.8. Let A be a JB-algebra with the 3-ball property. Then for each pure state $\rho \in \partial_e K$ there is associated at least one irreducible representation. A given irreducible representation π of A will be associated with ρ iff π is Jordan equivalent with φ_ρ . Furthermore, if an irreducible representation π of A is associated with $\rho \in \partial_e K$, then the set of pure states with which π is associated, is precisely $\partial_e F_\rho = F_\rho \cap \partial_e K$.

Proof. Let $\rho \in \partial_e K$. By Corollary 3.6, $A_\rho \cong B(H)_{sa}$, say that this isomorphism is effected by a Jordan isomorphism $\psi: A_\rho \rightarrow B(H)_{sa}$. Then $\psi \circ \varphi_\rho$ is a concrete representation of A which is Jordan equivalent to φ_ρ . Now let $\pi: A \rightarrow B(H)_{sa}$ be any irreducible representation which is Jordan equivalent to φ_ρ . By Proposition 2.6, π^* will be an affine isomorphism from the normal state space of $B(H)$ onto F_ρ . Since pure normal states on $B(H)$ are vector states, there exists a vector $\xi \in H$ satisfying (3.3). Hence π is associated with ρ .

If $\pi': A \rightarrow B(H')_{sa}$ is any irreducible representation associated with ρ , then $\tilde{\pi}'(c(\rho)) = I$, so $\ker \tilde{\pi}' = (e - c(\rho)) \circ A^{**}$. By Lemma 2.5, π' is Jordan equivalent to φ_ρ .

Finally, the set of pure states with which a given irreducible representation π is associated, will consist of precisely those $\sigma \in \partial_e K$ such that φ_σ is Jordan equivalent to φ_ρ ; by Proposition 2.6 this is equivalent to $\sigma \in F_\rho$. \square

The existence of irreducible representations associated with any given pure state of a JB-algebra with 3-ball property, follows from Proposition 3.8; but the uniqueness question is more complicated than for a C^* -algebra. In the Jordan algebra context one must allow for conjugate linear, as well as linear, isometries of the (complex) Hilbert spaces on which the algebras are represented. This will be explained in detail below, but first we recall some elementary facts on complex Hilbert spaces.

A typical example of a conjugate linear isometry of a complex Hilbert space H onto itself is obtained by considering an orthonormal basis $\{e_\alpha\}$ and then defining $J: H \rightarrow H$ by $J(\sum_\alpha \lambda_\alpha e_\alpha) = \sum_\alpha \overline{\lambda_\alpha} e_\alpha$. If J is defined in this way and if $U: H \rightarrow H'$ is unitary (i.e. a linear isometry onto), then UJ is a conjugate linear

isometry of H onto H' , and every conjugate linear isometry V of H onto H' is seen to be of this form. (Write $U = VJ$ and note that $UJ = V$ since $J^2 = I$). Note also that the transpose map $a \mapsto a^t$ of $B(H)$ onto itself with respect to the basis $\{e_\alpha\}$ is described by $a^t = Ja^*J$. Clearly $a \mapsto a^t$ is a *-anti-automorphism of $B(H)$, i.e. a linear map satisfying $(a^*)^t = (a^t)^*$ and $(ab)^t = b^t a^t$ for $a, b \in B(H)$.

It is well known that the *-isomorphisms of $B(H)$ onto $B(H')$ are precisely the maps

$$(3.4) \quad a \mapsto UaU^{-1},$$

where $U: H \rightarrow H'$ is unitary (see e.g. [15; Cor.III.3.1])

If ψ is a *-anti-isomorphism of $B(H)$ onto $B(H')$

and if $a \mapsto a^t$ is defined as above, then $a \mapsto \psi(a^t)$ is seen to be a *-isomorphism. Hence there exists a unitary map $U: H \rightarrow H'$ such that $\psi(a^t) = UaU^{-1}$ for all $a \in B(H)$. Writing $a^t = b$, we obtain $a = b^t$, and so $\psi(b) = Ub^tU^{-1}$. Thus the *-anti-isomorphisms of $B(H)$ onto $B(H')$ are precisely the maps

$$(3.5) \quad b \mapsto Ub^tU^{-1} = UJb^*JU^{-1},$$

where $U: H \rightarrow H'$ is unitary, and the map $J: H \rightarrow H$ and the transposition in $B(H)$ are defined with respect to an arbitrary orthonormal basis in H .

In (3.5) the map UJ is the most general form of a conjugate linear isometry, and we see that $(UJ)^{-1} = JU^{-1}$. Hence the above results can be assumed up as follows: The *-isomorphisms of $B(H)$ onto $B(H')$ are implemented by linear isometries of H onto H' , while the *-anti-isomorphisms are implemented by conjugate linear isometries of H onto H' .

Definition. Let A be a JB-algebra whose state space K has the 3-ball property, and let $\pi_j : A \rightarrow B(H)_{sa}$ ($j=1,2$) be two irreducible representations corresponding to the same $\rho \in \partial_e K$. We say that π_1 and π_2 are unitarily equivalent, respectively conjugate, if there exists a map W of H_1 onto H_2 which is unitary, respectively a conjugate linear isometry, such that

$$(3.6) \quad \pi_2(a) = W \pi_1(a) W^{-1}, \quad \text{all } a \in A.$$

Note that the only case in which π_1 and π_2 are both unitarily equivalent and conjugate at the same time, is when $\dim H_1 = \dim H_2 = 1$. (The map $\pi_1(a) \rightarrow \pi_2(a)$ extends uniquely to a strongly continuous linear map from $B(H_1)$ onto $B(H_2)$. If W is linear, the extension is $x \mapsto WxW^{-1}$, which is an isomorphism. If W is conjugate linear, the extension is $x \mapsto Wx^*W^{-1}$, which is an anti-isomorphism. The unique extension will be both an isomorphism and an anti-isomorphism iff $B(H_2)$ is commutative, or equivalently iff $\dim H_1 = \dim H_2 = 1$). Note also that to any given irreducible representation $\pi : A \rightarrow B(H)_{sa}$ one can associate a conjugate irreducible representation $\pi' : A \rightarrow B(H)_{sa}$ given by $\pi'(a) = \pi(a)^t$ where the transpose map is defined as above.

Proposition 3.9. Let A be a JB-algebra with the 3-ball property and let $\pi_i : A \rightarrow B(H_i)_{sa}$ ($i=1,2$) be two irreducible representations which are Jordan equivalent. Then π_1 and π_2 are either unitarily equivalent or conjugate or both; the last iff $\dim H_1 = \dim H_2 = 1$.

Proof. Let Φ be a Jordan isomorphism from $B(H_1)_{sa}$ onto $B(H_2)_{sa}$ with $\Phi \circ \pi_1 = \pi_2$. Let θ be the unique extension of Φ

to a complex linear Jordan isomorphism from $B(H_1)$ onto $B(H_2)$. By a known theorem [25] θ is either a $*$ -automorphism or a $*$ -anti-isomorphism. Then it follows from the remarks above that θ is implemented by a map W of H_1 onto H_2 which is either unitary or a conjugate linear isometry. Since $\pi_2 = \theta \circ \pi_1$, we have $\pi_2(a) = W\pi_1(a)W^{-1}$ for each $a \in A$. \square

The following lemma provides a useful characterization of one-dimensional representations.

Lemma 3.10. Let A be a JB-algebra with the 3-ball property, and $\pi: A \rightarrow B(H)_{sa}$ an irreducible representation. Then $\dim H = 1$ iff for some (then all) non-zero vectors $\xi \in H$:

$$(3.7) \quad ([\pi(a), \pi(b)]\xi | \xi) = 0 \quad \text{for all } a, b \in A.$$

Proof. If $\dim H = 1$, then (3.7) is clear. Conversely, if (3.7) holds, then by density of $\pi(A)$ in $B(H)_{sa}$:

$$([s, t]\xi | \xi) = 0 \quad \text{for all } s, t \in B(H).$$

If $\dim H > 1$, we could contradict this by choosing $s, t \in B(H)$ such that $s\xi = t\xi = \eta \notin \text{lin}(\xi)$ and $s\eta = -t\eta = \xi$, which would give $([s, t]\xi | \xi) = 2\|\xi\|^2 \neq 0$. \square

Proposition 3.11. Let A be a JB-algebra whose state space K has the 3-ball property and let $\pi_j: A \rightarrow B(H_j)_{sa}$ ($j = 1, 2$) be two irreducible representations corresponding to the same $\rho \in \partial_e K$. Suppose also that $\xi_1 \in H_1$ and $\xi_2 \in H_2$ are two unit vectors representing ρ (with respect to π_1 and π_2 respectively). Then π_1 and π_2 are unitarily equivalent iff for all $a, b \in A$:

$$(3.8) \quad ([\pi_2(a), \pi_2(b)]\xi_2 | \xi_2) = ([\pi_1(a), \pi_1(b)]\xi_1 | \xi_1) ,$$

and π_1 and π_2 are conjugate iff for all $a, b \in A$

$$(3.9) \quad ([\pi_2(a), \pi_2(b)]\xi_2 | \xi_2) = -([\pi_1(a), \pi_1(b)]\xi_1 | \xi_1) .$$

Proof. By Proposition 2.8 and Lemma 3.10, it suffices to establish (3.8) when π_1 and π_2 are unitarily equivalent and (3.9) when they are conjugate. We only present the latter argument; the former is similar (just simpler).

Thus we assume that π_1 and π_2 are conjugate, and choose a conjugate linear isometry W satisfying (3.6). Note that for all $\xi \in H_1$, $\eta \in H_2$ one has $(W\xi | \eta) = (\xi | W^{-1}\eta)^{-}$. Thus for every $a \in A$:

$$\begin{aligned} (\pi_1(a)\xi_1 | \xi_1) &= \langle a, \rho \rangle = (\pi_2(a)\xi_2 | \xi_2) \\ &= (W\pi_1(a)W^{-1}\xi_2 | \xi_2) = (\pi_1(a)W^{-1}\xi_2 | W^{-1}\xi_2) . \end{aligned}$$

Hence $W^{-1}\xi_2 = \lambda\xi_1$ where $|\lambda| = 1$. Now

$$\begin{aligned} ([\pi_2(a), \pi_2(b)]\xi_2 | \xi_2) &= (W[\pi_1(a), \pi_1(b)]W^{-1}\xi_2 | \xi_2) \\ &= \lambda\overline{\lambda}([\pi_1(a), \pi_1(b)]\xi_1 | \xi_1)^{-} = -([\pi_1(a), \pi_2(b)]\xi_2 | \xi_2) . \quad \square \end{aligned}$$

The following will be of use later.

Proposition 3.12. Let A be a JB-algebra with the 3-ball property. If $\pi : A \rightarrow B(H)_{sa}$ is an irreducible representation, then the pure states with which π is associated are precisely those of the form $a \mapsto (\pi(a)\xi | \xi)$ where $\xi \in H$, $\|\xi\| = 1$.

Proof. By Proposition 3.7 and Proposition 2.6, π^* maps the normal state space of $B(H)$ (affine) isomorphically onto F_ρ for some pure state $\rho \in \partial_e K$. Each vector state on $B(H)$ is a pure normal state, and so it is mapped by π^* into $\partial_e F_\rho \subseteq \partial_e K$. \square

§ 4. Reversibility.

Following Størmer [34; p.439] we will say that a JC-algebra A is reversible if

$$(4.1) \quad a_1 a_2 \dots a_n + a_n a_{n-1} \dots a_1 \in A$$

whenever $a_1, \dots, a_n \in A$. Note that the left hand side of (4.1) is the Jordan triple product for $n=3$. Thus, (4.1) always holds for $n=3$, but it is worth noting that it can fail already for $n=4$. (In fact, $n=4$ is the critical value; if (4.1) holds for $n=4$, then it holds for all $n>4$ as shown by P.M. Cohn [13]).

For a given JC-algebra $A \subseteq B(H)_{sa}$ we denote by $\mathcal{R}_0(A)$ the real subalgebra of $B(H)$ generated by A , and we denote by $\mathcal{R}(A)$ the norm closure of $\mathcal{R}_0(A)$. We observe that $\mathcal{R}_0(A)$ is closed under the $*$ -operation since $(a_1 a_2 \dots a_n)^* = a_n a_{n-1} \dots a_1$ for $a_1, \dots, a_n \in A$. From this it follows that $\mathcal{R}(A)$ is a norm closed real $*$ -algebra of operators on H . (Such an algebra is sometimes called a "real C*-algebra"). If A is reversible and $b = a_1 a_2 \dots a_n$ where $a_1, \dots, a_n \in A$, then the self-adjoint part $b_h = \frac{1}{2}(b+b^*)$ will be in A . From this it follows that A is reversible iff $\mathcal{R}_0(A)_{sa} = A$.

Assume now that A is reversible and consider an element $b \in \mathcal{R}(A)_{sa}$, say $b = b^*$ and $b = \lim_n b_n$ where $b_n \in \mathcal{R}_0(A)$ for $n=1, 2, \dots$ (norm limit). Then $b = \lim_n (b_n)_h \in A$ since A is closed. From this it follows that A is reversible iff $\mathcal{R}(A)_{sa} = A$.

By definition, reversibility is a spatial notion involving the non-commutative multiplication of Hilbert space operators. In general it is not an isomorphism invariant; it is possible for a reversible and a non-reversible JC-algebra to be isomorphic. This situation is illustrated by the spin-factors (definition below). A spin factor $A \subseteq B(H)_{sa}$ is always reversible when

$\dim A = 3$ or 4 , non-reversible when $\dim A \neq 3, 4$ or 6 , and it can be either reversible or non-reversible when $\dim A = 6$, even though all spin factors of the same dimension are isomorphic. Of these results we will prove only the one with $\dim A = 4$, since we shall not need the others.

Recall that an (abstract) spin factor is a JB-algebra S of dimension at least three which is a real Hilbert space such that $(e|e) = 1$ and $s \circ t = (s|t)e$ for each pair of elements s, t in the hyperplane $N = \{e\}^\perp$. (Note that this requirement completely determines the Jordan product in S). Recall also that the Hilbert norm of a spin factor is equivalent with the JB-algebra norm, and that the two coincide on N (cf. [39]). It follows that every spin factor is a Banach dual space, hence a JBW-algebra. It is easily verified that the center of any spin factor is trivial, hence it is a factor (which justifies the terminology). In fact, the spin factors are precisely the JBW-factors of type I_2 (see [7; § 7] for definition and proof).

If S is a spin factor, then the hyperplane $N = \{e\}^\perp$ consists of all elements λs where $\lambda \in \mathbb{R}$, $s \neq e$, and s is a symmetry, i.e. $s^2 = e$. Note also that two elements of N are orthogonal iff their Jordan product is zero. Thus, if $\{s_\alpha\}$ is an ortho-normal basis in S such that $s_{\alpha_0} = e$ for some index α_0 , then all the other basis-elements are symmetries satisfying $s_\alpha \circ s_\beta = \delta_{\alpha, \beta} e$. For later references we observe that the orthogonal components of an element $a \in S$ with respect to such a basis, can be expressed in terms of the Jordan product. In fact, if $a = \lambda_0 e + \sum_{\alpha \neq \alpha_0} \lambda_\alpha s_\alpha$, then for each $\alpha \neq \alpha_0$:

$$(a \circ s_\alpha) \circ s_\alpha = (\lambda_0 s_\alpha + \lambda_\alpha e) \circ s_\alpha = \lambda_0 e + \lambda_\alpha s_\alpha ;$$

hence for any index $\beta \neq \alpha_0$ distinct from α_0 :

$$(4.2) \quad (((a \circ s_\alpha) \circ s_\alpha) \circ s_\beta) \circ s_\beta = \lambda_0 e ;$$

moreover:

$$(4.3) \quad (a - \lambda_0 e) \circ s_\alpha = \lambda_\alpha e .$$

Simple examples of spin factors are the Jordan algebra $M_2(\mathbb{R})_s$ of all symmetric 2×2 -matrices over \mathbb{R} and the Jordan algebra $M_2(\mathbb{C})_{sa}$ of all self-adjoint 2×2 -matrices over \mathbb{C} . For these algebras, ortho-normal bases are respectively (s_0, s_1, s_2) and (s_0, s_1, s_2, s_3) , where s_0 is the unit matrix and s_1, s_2, s_3 are the elementary spin matrices:

$$(4.4) \quad s_1 = \begin{Bmatrix} 1 & 0 \\ 0 & -1 \end{Bmatrix}, \quad s_2 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}, \quad s_3 = \begin{Bmatrix} 0 & i \\ -i & 0 \end{Bmatrix} .$$

It follows from the above discussion that two spin factors of the same dimension must be isomorphic. In particular, every spin factor of dimension three is isomorphic to $M_2(\mathbb{R})_s$, and every spin factor of dimension four is isomorphic to $M_2(\mathbb{C})_{sa}$.

Lemma 4.1. Every Jordan homomorphism $\varphi: M_2(\mathbb{C})_{sa} \rightarrow B(H)_{sa}$ extends uniquely to a real $*$ -homomorphism $\tilde{\varphi}: M_2(\mathbb{C}) \rightarrow B(H)$. Consequently, if $A \subseteq B(H)_{sa}$ and A is a four-dimensional spin factor, then $\mathcal{R}(A)$ is isomorphic (as a real $*$ -algebra) to $M_2(\mathbb{C})$, and A is reversible.

Proof. Considered as a real linear space, $M_2(\mathbb{C})$ has a basis consisting of the eight elements $s_0, s_1, s_2, s_3, is_0 = -s_1s_2s_3, is_1 = -s_2s_3, is_2 = -s_3s_1, is_3 = -s_1s_2$. Hence the only possible candidate for $\tilde{\varphi}$ is

$$\begin{aligned} & \tilde{\varphi}\left(\sum_{j=0}^3 \alpha_j s_j + \sum_{j=0}^3 \beta_j (is_j)\right) \\ &= \sum_{j=0}^3 \alpha_j \varphi(s_j) - \beta_0 \varphi(s_1) \varphi(s_2) \varphi(s_3) - \sum_{(j,k,l)} \beta_j \varphi(s_k) \varphi(s_l), \end{aligned}$$

where (j,k,l) runs through the three cyclic permutations of $(1,2,3)$. It is now straightforward to verify that $\tilde{\varphi}$ actually is a real $*$ -homomorphism of $M_2(\mathbb{C})$ into $B(H)$. (Note that the relations $s_j \circ s_k = \delta_{jk} s_0$ and $s_j \circ s_k = -s_k \circ s_j$ where $j,k=1,2,3$, are preserved by the Jordan homomorphism φ).

Now if $A \subseteq B(H)_{sa}$ is any four dimensional spin factor, let $\varphi: M_2(\mathbb{C})_{sa} \rightarrow A$ be an isomorphism. Note that $\tilde{\varphi}$ will map $M_2(\mathbb{C}) = (M_2(\mathbb{C})_{sa})$ onto $\mathcal{R}(A)$ and will be a real $*$ -isomorphism from $M_2(\mathbb{C})$ onto $\mathcal{R}(A)$. Finally, reversibility of $M_2(\mathbb{C})_{sa}$ in $M_2(\mathbb{C})$ implies reversibility of $A \subseteq \mathcal{R}(A) \subseteq B(H)$. \square

We will reduce the problem of reversibility for a given JC-algebra to the same problem for its weak closure in an appropriate representation. Then we are in a setting where the structure theory for JW-algebras applies. Recall in this connection that any given JW-algebra $A \subseteq B(H)_{sa}$ can be written as

$$(4.5) \quad A = A_1 \oplus A_2 \oplus \dots \oplus A_\infty \oplus B,$$

where A_1 is an abelian JW-algebra, A_j is of type I_j for $j = 2, 3, \dots, \infty$, and B is the non-type I summand. (See [39; Ths. 5 & 16] for precise definitions and proofs, but note in particular that the direct sum (3.4) is given by orthogonal central idempotents $z_1, z_2, \dots, z_\infty$, $w \in A$ such that $z_j A = A_j$ for $j = 1, 2, \dots, \infty$ and $wA = B$).

We will see later that the I_2 -summand is the key to reversibility. Therefore we will now study JW-algebras of type I_2 . We begin by two technical lemmas.

Lemma 4.2. For each integer $n \geq 1$ there exists a Jordan polynomial P_n in $n+2$ variables such that for any spin factor S and an arbitrary pair s, t of orthogonal symmetries in S we have $P_n(s, t, a_1, \dots, a_n) = 0$ iff $a_1, \dots, a_n \in S$ are linearly dependent.

Proof. By the well known Gramm criterion for spaces with an inner product, n elements a_1, \dots, a_n of a spin factor S will be linearly dependent iff $\text{Det}\{(a_i | a_j)\}_{i,j=1}^n = 0$. Since the Jordan multiplication in S reduces to scalar multiplication in $\text{Re} \subseteq S$, we can rewrite this condition as

$$(4.6) \quad Q_n((a_1 | a_1)e, (a_1 | a_2)e, \dots, (a_n | a_n)e) = 0,$$

where Q_n is an appropriate Jordan polynomial in n^2 variables.

Assume now that s, t are two arbitrary (but fixed) orthogonal symmetries in S . For any set (a_1, \dots, a_n) of n elements of S we decompose each a_j as $a_j = \alpha_j e + n_j$ where $n_j \in N = \{e\}^\perp$. For given i, j the multiplication rules for spin factors give:

$$\begin{aligned} (a_i | a_j)e &= \alpha_i \alpha_j e + (n_i | n_j)e \\ &= (\alpha_i e) \circ (\alpha_j e) + n_i \circ n_j = (\alpha_i e) \circ (\alpha_j e) + (a_i - \alpha_i e) \circ (a_j - \alpha_j e). \end{aligned}$$

It follows from (4.2) that $(a_i | a_j)e$ can be expressed as a Jordan polynomial in s, t, a_i, a_j for $i, j = 1, 2, \dots, n$. Substituting these polynomials into Q_n , we obtain a Jordan polynomial P_n in the $n+2$ variables s, t, a_1, \dots, a_n , which will have the desired property. Clearly, P_n is independent of the spin factor S and the choice of s and t . \square

Observe for later applications that if A is a JW-algebra of type I_2 and if $\varphi \rightarrow M$ is a dense representation, then M must be a spin factor. In fact, if p and q are exchangeable

abelian projections in A with sum 1, then $\varphi(p)$ and $\varphi(q)$ are exchangeable abelian projections in M with sum 1, so M is an I_2 -factor, i.e. a spin factor.

For the next lemma we also need some new terminology: Two elements a, b of a JB-algebra are said to be J-orthogonal if $a \cdot b = 0$. Clearly this generalizes the orthogonality of symmetries in a spin factor. Note also that if A is concretely represented as a JC-algebra, then a, b are J-orthogonal iff the operator ab is skew. For a given idempotent p in a JB-algebra A we say that an element $s \in A$ is a p-symmetry if $s^2 = p$.

Lemma 4.3. If a projection p in a JW-algebra A of type I_2 admits two J-orthogonal p-symmetries, then p is central.

Proof. Let s, t be two J-orthogonal p-symmetries in A , and define $q = \frac{1}{2}(p+s)$, $r = \frac{1}{2}(p-s)$. Then $q+r = p$, and q, r are exchangeable projections; in fact the symmetry $u = (1-p)+t$ satisfies $uqu = r$, so it exchanges q and r .

Note that the central covers $c(p), c(q), c(r)$ are all equal. We assume for contradiction that $p \neq c(p)$. Then the central covers of q and of $c(p)-p$ will not be orthogonal, so by [37; Lem.18] there will exist exchangeable non-zero projections $x \leq q$, $y \leq c(p)-p$. Defining $z = uxu$, we get $z \leq uqu = r$.

Now x, y, z are non-zero orthogonal projections with x, y exchangeable and x, z exchangeable. Then any homomorphism which annihilates one of the projections x, y, z , will annihilate the other two. Thus there exists a dense representation $\varphi: A \rightarrow M$ which does not annihilate any of the three projections x, y, z (cf. [7; Cor.5.7]). By the remark preceding this lemma, M must

be a spin factor. But a spin factor cannot contain a set of three non-zero orthogonal projections. (A non-trivial projection u in a spin factor is minimal and admits just one orthogonal projection, namely $e-u$). This contradiction completes the proof. \square

The next lemma is crucial.

Lemma 4.4. If $A \subseteq B(H)_{sa}$ is a JC-algebra whose state space has the 3-ball property, then every dense representation of the I_2 -summand of \bar{A} is onto a spin factor of dimension at most four.

Proof. Let z be the central projection in \bar{A} such that the I_2 -summand of \bar{A} is equal to $z\bar{A}$, and let $\varphi: z\bar{A} \rightarrow M$ be a dense representation. As remarked earlier, M must be a spin factor.

Note that $M_0 = \varphi(zA)$ will be a norm closed Jordan subalgebra of M containing the identity. It is not difficult to verify that such a subalgebra is itself a spin factor unless it is of dimension less than three. In the latter case M_0 will be associative (in fact $M \cong \mathbb{R}$ or $M \cong \mathbb{R} \oplus \mathbb{R}$). In the former case it follows from Proposition 3.7 that $M_0 \cong B(H_0)_{sa}$ for some finite or infinite Hilbert space H_0 ; but $B(H_0)_{sa}$ is a spin factor only if H_0 is of (complex) dimension 2, in which case $B(H_0)_{sa}$ is of (real) dimension 4. Hence $\dim M_0 = 1, 2$ or 4 .

We will next show that $\dim M \leq 4$. Let p, q be exchangeable abelian projections in $z\bar{A}$ with $p+q = z$. Then there exists a z -symmetry $s \in z\bar{A}$ such that $sps = q$. Now $ps = sq$, so $s(p-q) = (q-p)s$. Thus the elements s and $t = p-q$ are symmetries in the Jordan algebra $z\bar{A}$ satisfying $s \circ t = 0$. Consider now an arbitrary dense representation ψ of $z\bar{A}$. By the above

argument (with ψ in place of φ), ψ is a spin factor representation and $\dim \psi(zA) \leq 4$. By Lemma 3.3 we have

$$\begin{aligned} & \psi(P_5(s, t, za_1, \dots, za_5)) \\ &= P_5(\psi(s), \psi(t), \psi(za_1), \dots, \psi(za_5)) = 0 \end{aligned}$$

for any set of five elements $a_1, \dots, a_5 \in A$. Since the dense representations separate points [7; Cor.5.7 and Prop.8.7], it follows that

$$(4.7) \quad P_5(s, t, za_1, \dots, za_5) = 0, \quad \text{all } a_1, \dots, a_5 \in A.$$

By the Kaplansky density theorem for JC-algebras [18; p.314], the unit ball of zA is strongly dense in the unit ball of $z\bar{A}$. Hence it follows from (4.7) that $P_5(s, t, x_1, \dots, x_5) = 0$ for all $x_1, \dots, x_5 \in z\bar{A}$. Applying φ , we get

$$P_5(\varphi(s), \varphi(t), \varphi(x_1), \dots, \varphi(x_5)) = 0 \quad \text{all } x_1, \dots, x_5 \in z\bar{A}.$$

By Lemma 3.3, $\varphi(x_1), \dots, \varphi(x_5)$ is a linearly dependent set of elements of M for any set of five elements $x_1, \dots, x_5 \in z\bar{A}$. Hence $\dim \varphi(z\bar{A}) \leq 4$, and by σ -weak density, $\dim M \leq 4$. \square

It follows from the next result that the dense spin factor representations of Lemma 4.4 have dimension precisely four.

Lemma 4.5. Let $A \subseteq B(H)_{sa}$ be a JC-algebra whose state space has the 3-ball property, and let s_0 be the central projection in \bar{A} such that $s_0\bar{A}$ is the I_2 -summand of \bar{A} . Then $s_0\bar{A}$ contains a subalgebra $M = \text{lin}_{\mathbb{R}}(s_0, s_1, s_2, s_3)$ which is a four dimensional spin factor with s_1, s_2, s_3 J-orthogonal s_0 -symmetries. Moreover, each $b \in s_0\bar{A}$ can be uniquely expressed as:

$$(4.8) \quad b = \sum_{j=0}^3 f_j s_j ,$$

where f_j is in the center Z of $s_0 \bar{A}$ for $j = 0, 1, 2, 3$.

Proof. Let $\{p_\alpha\}$ be a maximal orthogonal set of central projections in $s_0 \bar{A}$ with the property that each p_α admits three J-orthogonal p_α -symmetries, say $s_{1\alpha}, s_{2\alpha}, s_{3\alpha}$ and let $p = \sum_\alpha p_\alpha$. A priori, there may not exist any such p_α , in which case the summation over the empty set of indices would give $p = 0$. However, we shall see that this eventuality cannot occur; in fact we will prove that $p = s_0$.

Assume that $p \neq s_0$. Now we will first show that every dense representation of $(s_0 - p)\bar{A}$ is of dimension at most three, then we will see that this leads to a contradiction. By Lemma 4.4 all dense representations of $(s_0 - p)\bar{A}$ are onto spin factors of dimension at most four (since each extends to the I_2 -summand of \bar{A}). Now if $\varphi: (s_0 - p)\bar{A} \rightarrow M$ is a four-dimensional spin factor representation, then by [34; Lem.3.6] we can find orthogonal symmetries s_1, s_2, s_3 in M , an idempotent $q \in (s_0 - p)\bar{A}$, and J-orthogonal q -symmetries t_1, t_2, t_3 mapping onto s_1, s_2, s_3 , respectively. Note that $(s_0 - p)\bar{A}$ is of type I_2 , and so by Lemma 4.3, q is a central idempotent. This contradicts the maximality of $\{p_\alpha\}$, so we conclude that all dense representations of $(s_0 - p)\bar{A}$ are onto spin factors of dimension three. Now, as in the proof of Lemma 4.4, all such representations restricted to $(s_0 - p)\bar{A}$ have associative range. Thus $(s_0 - p)\bar{A}$ must be associative (i.e. abelian). But this is impossible, so $p = s_0$ as claimed.

Define $s_j = \sum_\alpha s_{j\alpha}$ for $j = 1, 2, 3$. Then s_1, s_2, s_3 are J-orthogonal s_0 -symmetries, and $M = \text{lin}_{\mathbb{R}}(s_0, s_1, s_2, s_3)$ is a

spin factor of dimension four.

It remains to establish the decomposition (4.8). For a given $b \in s_0 \bar{A}$ we define

$$(4.9) \quad f_0 = (((b \circ s_1) \circ s_1) \circ s_2) \circ s_2 ,$$

$$(4.10) \quad f_j = (b - f_0) \circ s_j , \quad \text{for } j = 1, 2, 3 .$$

Consider now a dense representation ψ of $s_0 \bar{A}$. Since ψ is a dense spin factor representation of dimension at most four (by Lemma 4.4), and since $\psi(s_1), \psi(s_2), \psi(s_3)$ are orthogonal symmetries, we have a decomposition

$$\psi(b) = \sum_{j=0}^3 \lambda_j \psi(s_j) = \sum_{j=0}^3 (\lambda_j e) \circ \psi(s_j) ,$$

where the coefficients λ_j are given as in the formulas (4.2) and (4.3). Comparing these formulas with (4.9) and (4.10) (with $a = \psi(b)$), we conclude that

$$(4.11) \quad \lambda_j e = \psi(f_j) , \quad \text{for } j = 0, 1, 2, 3 ,$$

and therefore

$$(4.12) \quad \psi(b) = \psi\left(\sum_{j=0}^3 f_j \circ s_j\right) .$$

By (4.11) and (4.12), ψ will map the elements f_0, f_1, f_2, f_3 onto central elements and the element $b - \sum_{j=0}^3 f_j \circ s_j$ onto zero. Since the dense representations separate points, it follows that $f_0, f_1, f_2, f_3 \in Z$ and that $b - \sum_{j=0}^3 f_j \circ s_j = 0$. The uniqueness follows from (4.9) and (4.10). \square

Remark. Note that Lemma 4.5, equation (4.8), implies that the I_2 -summand of \bar{A} is isomorphic to $C(X, M_2(\mathbb{C})_{sa})$ where X is a hyperstonean space such that $C(X)$ is isomorphic to the center of $s_0 \bar{A}$.

The next theorem is the main result of this section.

Theorem 4.6. If the state space of a JC-algebra $A \subseteq B(H)_{sa}$ has the 3-ball property, then A is reversible.

Proof. 1.) Let \mathcal{O} be the C^* -algebra generated by A , and let $\pi: \mathcal{O} \rightarrow B(H')$ be the universal representation of \mathcal{O} . Since reversibility of A only depends on the embedding of A in \mathcal{O} , we can, and shall, identify \mathcal{O} and $\pi(\mathcal{O})$. First we will show that \bar{A} is reversible in this representation.

By [36; Th.6.4. & Th.6.6] it suffices to show that the I_2 -summand of \bar{A} is reversible. Let this summand be $s_0 \bar{A}$ where s_0 is a central projection in \bar{A} , and let s_1, s_2, s_3 be J -orthogonal s_0 -symmetries with the properties explained in Lemma 4.5. Consider now an arbitrary finite set of elements

$$b_i = \sum_{j=0}^3 f_{ij} s_j \in s_0 \bar{A}, \quad i = 1, \dots, n,$$

where the coefficients f_{ij} are in the center of $s_0 \bar{A}$. By Lemma 3.1 the spin factor $M = \text{lin}_{\mathbb{R}}(s_0, s_1, s_2, s_3)$ is reversible. Hence

$$b_1 \cdots b_n + b_n \cdots b_1 = \sum_{(j_1, \dots, j_n)} f_{1j_1} \cdots f_{nj_n} (s_{j_1} \cdots s_{j_n} + s_{j_n} \cdots s_{j_1}) \in s_0 \bar{A}.$$

This shows that $s_0 \bar{A}$ is reversible, and thus \bar{A} is reversible.

2.) We now show that A is reversible. Suppose $a_1, \dots, a_n \in A$; by reversibility of \bar{A}

$$x = a_1 a_2 \cdots a_n + a_n \cdots a_{n-1} a_1 \in \bar{A}.$$

But x is also in \mathcal{O} , and so it lies in $\mathcal{O} \cap \bar{A}$. We are done if we show $\mathcal{O} \cap \bar{A} = A$.

Recall that $\overline{\mathcal{O}}$ can be identified with \mathcal{O}^{**} . The weak and σ -weak closures of A will coincide [37; Lem.4.2], so \overline{A} is also the σ -weak closure of A (i.e. the closure in $w(\mathcal{O}^{**}, \mathcal{O}^*)$). Now $\overline{A} \cap \mathcal{O}$ is obtained by intersecting \mathcal{O} with the intersecting of all $w(\mathcal{O}^{**}, \mathcal{O}^*)$ -closed hyperplanes containing A ; but these hyperplanes are of the form $\varphi^{-1}(0)$ where $\varphi \in \mathcal{O}^*$, and thus $\overline{A} \cap \mathcal{O} = A$ since A is norm closed. This completes the proof. \square

Remark. In the above proof the theorems 6.4 and 6.6 of [36] were used in an essential way. However, one does not really need the full development leading up to these theorems. What is relevant for reversibility, is that the identity element of each of the direct summands A_3, \dots, A_∞ and B of (4.4) is the sum of a finite number of exchangeable idempotents, from this it follows that each of the real $*$ -algebras $\mathcal{R}(A_3), \dots, \mathcal{R}(A_\infty)$ and $\mathcal{R}(B)$ contains a set of matrix units $\{e_{ij}\}$ with $\{e_{ij} + e_{ji}\}$ contained in A_3, \dots, A_∞ and B , respectively. Therefore it is expressible as a matrix algebra over an appropriate associative algebra, and one can show that the self-adjoint part of this matrix algebra is A_3, \dots, A_∞ and B , respectively. Now the reversibility follows.

§ 5. The enveloping C*-algebra.

In this section we will show that if a JB-algebra A has a state space with the 3-ball property, then there exists an "enveloping C*-algebra" \mathcal{O} and an embedding $\psi: A \rightarrow \mathcal{O}$ with the universal property that every Jordan homomorphism $\pi: A \rightarrow B(H)_{sa}$ extends uniquely to a *-homomorphism $\tilde{\pi}: \mathcal{O} \rightarrow B(H)$; moreover, the pair consisting of the C*-algebra \mathcal{O} and the embedding ψ is unique in the natural sense of the word.

It follows from [6; Prop.7.6] that every JB-algebra whose state space has the 3-ball property, can be concretely represented as a JC-algebra $A \subseteq B(H_0)_{sa}$ for some Hilbert space H_0 . One might expect that the enveloping C*-algebra of such a concretely represented algebra A should be the C*-algebra generated by A in $B(H_0)$, but in general this is not the case. (The transpose map from $A = M_n(\mathbb{C})_{sa}$ to $M_n(\mathbb{C})_{sa}$ does not extend to a *-homomorphism from the generated C*-algebra $M_n(\mathbb{C})$ to $M_n(\mathbb{C})$). In fact, the C*-algebra generated by a JC-algebra $A \subseteq B(H_0)_{sa}$ is not an isomorphism invariant, while the enveloping C*-algebra is unique up to isomorphisms, as mentioned above.

We will first show that for a given JC-algebra $A \subseteq B(H_0)_{sa}$, then the real *-algebra $\mathcal{R}(A)$, unlike the generated C*-algebra, will be unique up to *-isomorphisms. This result will be used in an essential way in the proof of the universal property of the enveloping C*-algebra.

Lemma 5.1. Let A be a JW-algebra which is either homogeneous of type I_n with $n \neq 2$, or has no type I part. Then every Jordan homomorphism $\varphi: A \rightarrow B(H)_{sa}$ extends uniquely to a real *-homomorphism $\tilde{\varphi}: \mathcal{R}(M) \rightarrow B(H)$.

Proof. If A is of type I_1 , then A is abelian and the result is evident. If A is homogeneous of type I_n with $n \geq 3$, or if A has no type I part, then by definition in the former case and by [39; Th.17] in the latter we can write $e = \sum_{\alpha} p_{\alpha}$ where $\{p_{\alpha}\}$ are exchangeable projections and the cardinality of $\{p_{\alpha}\}$ is at least three. By [39; Th.9] we can group together projections (if necessary) so that e can be written as a sum of m exchangeable projections, say p_1, \dots, p_m , with $3 \leq m < \infty$. Let s_1, \dots, s_m be symmetries in A which exchange p_1 and p_j , i.e. $s_j p_1 s_j = p_j$ for $1 \leq j \leq m$. Define $e_{ij} = p_i s_i p_1 s_j p_j$, and note that $\{e_{ij}\}$ is a set of matrix units for $\mathcal{R}(A)$, i.e. $e_{ij} = \delta_{jk} e_{il}$, $e_{ij}^* = e_{ji}$, $\sum_i e_{ii} = e$. Thus $\mathcal{R}(A)$ can be expressed as a $m \times m$ matrix algebra, say $\mathcal{R}(A) = M_m(B)$ where B is a real $*$ -algebra. By [35; Th.4.6] A is reversible, so $A = \mathcal{R}(A)_{sa} = M_m(B)_{sa}$. It follows that A generates $\mathcal{R}(A)$ algebraically, i.e. $\mathcal{R}_0(A) = \mathcal{R}(A)$. Now by [22; Th.4] φ extends uniquely to a real homomorphism $\tilde{\varphi}: \mathcal{R}(A) \rightarrow B(H)$. Furthermore, for $a_1, \dots, a_n \in A$, then

$$\begin{aligned} \tilde{\varphi}((a_1 a_2 \cdots a_n)^*) &= \tilde{\varphi}(a_n a_{n-1} \cdots a_1) = \tilde{\varphi}(a_n) \tilde{\varphi}(a_{n-1}) \cdots \tilde{\varphi}(a_1) \\ &= (\tilde{\varphi}(a_1) \tilde{\varphi}(a_2) \cdots \tilde{\varphi}(a_n))^* = \tilde{\varphi}(a_1 a_2 \cdots a_n)^* ; \end{aligned}$$

thus $\tilde{\varphi}$ is a real $*$ -homomorphism. \square

Theorem 4.2. Let $A \subseteq B(H_0)_{sa}$ be a JC-algebra whose state space has the 3-ball property. Then every Jordan homomorphism $\varphi: A \rightarrow B(H)_{sa}$ admits a unique extension $\tilde{\varphi}: \mathcal{R}(A) \rightarrow B(H)$ where $\tilde{\varphi}$ is a real $*$ -homomorphism. Moreover, $\tilde{\varphi}$ will map $\mathcal{R}(A)$ into $\mathcal{R}(\varphi(A))$, and if φ is 1-1, then $\tilde{\varphi}$ will be a real $*$ -isomorphism from $\mathcal{R}(A)$ onto $\mathcal{R}(\varphi(A))$.

Proof. Let \mathcal{O} be the C^* -algebra generated by A in $B(H_0)$. Now let $\pi: \mathcal{O} \rightarrow B(H)$ be the universal representation of \mathcal{O} ; we will identify \mathcal{O} and $\pi(\mathcal{O})$. Then we can identify the σ -weak closure $\bar{\mathcal{O}}$ equipped with the σ -weak topology with \mathcal{O}^{**} equipped with the w^* -topology; furthermore we can identify $\bar{\mathcal{O}}_{sa}$ with \mathcal{O}_{sa}^{**} and \bar{A} with A^{**} (cf. [16; § 12] and [18]). Now it follows by Lemma 2.3 that we can extend a given Jordan homomorphism $\varphi: A \rightarrow B(H)_{sa}$ to a σ -weakly continuous Jordan homomorphism $\bar{\varphi}: \bar{A} \rightarrow B(H)_{sa}$.

As in (4.5) we have a decomposition

$$(5.1) \quad \bar{A} = A_1 \oplus \dots \oplus A_\infty \oplus B,$$

where $\bar{A}_n = z_n \bar{A}$ for $n = 1, 2, \dots, \infty$ and $B = w \bar{A}$ for central idempotents $z_1, z_2, \dots, z_\infty, w$ summing to the identity element. It follows from Lemma 5.1 that the restriction of $\bar{\varphi}$ to each of the summands $A_1, A_3, \dots, A_\infty$ and B can be extended to a real $*$ -homomorphism into $B(H)$ from $\mathcal{R}(A_1)$, $\mathcal{R}(A_3), \dots, \mathcal{R}(A_\infty)$ and $\mathcal{R}(B)$, respectively.

The I_2 -summand A_2 requires a separate treatment. We consider a spin factor $S = \text{lin}_{\mathbb{R}}(s_0, s_1, s_2, s_3) \subseteq A$ ($s_0 = z_2$) with the properties mentioned in Lemma 4.5. Then the elements of A_2 are of the form

$$(5.2) \quad b = \sum_{i=0}^3 f_i s_i,$$

where f_0, f_1, f_2, f_3 are in the center Z of A_2 .

By Lemma 4.1 $\mathcal{R}(S) \cong M_2(\mathbb{C})$, and by the proof of this lemma $\mathcal{R}(S)$ has a basis $\{m_0, \dots, m_7\}$ over \mathbb{R} such that $m_k = s_k$ and $m_{k+4} = js_k$ for $k = 0, 1, 2, 3$, with j some central element of $\mathcal{R}(S)$ such that $j^2 = -e$ and $j^* = -j$. Note that if we multiply together elements of the form

$$(5.3) \quad c = \sum_{i=0}^7 f_i m_i ,$$

where f_0, f_1, \dots, f_7 are in the center of A_2 , then we get an element of the same form (since each product $m_i m_k \in \mathcal{R}(S)$ is expressible in this form). By Lemma 4.5 the representation (5.2) of elements of A_2 is unique; from this it now follows that the elements of $\mathcal{R}(A_2)$ can be uniquely represented in the form (5.3).

Now we can extend $\bar{\varphi}|_S$ to a real $*$ -homomorphism $\tilde{\varphi}_0: \mathcal{R}(S) \rightarrow B(H)$ by Lemma 4.1, and then to a real $*$ -homomorphism $\tilde{\varphi}: \mathcal{R}(A_2) \rightarrow B(H)$ by writing

$$\tilde{\varphi}\left(\sum_{i=0}^7 f_i m_i\right) = \sum_{i=0}^7 \bar{\varphi}(f_i) \tilde{\varphi}_0(m_i) .$$

(The only non-trivial point in the verification that $\tilde{\varphi}$ is a $*$ -homomorphism is to observe that a Jordan homomorphism maps commuting operators into commuting operators, cf. [7; Lem.5.2]).

For an arbitrary $a \in \mathcal{R}(\bar{A})$ we define $\tilde{\varphi}(a)$ by applying the extended $*$ -homomorphisms $\tilde{\varphi}$ already defined on each summand.

Thus we write

$$\tilde{\varphi}(a) = \tilde{\varphi}(az_1) + \dots + \tilde{\varphi}(az_\infty) + \tilde{\varphi}(aw)$$

for every $a \in \mathcal{R}(\bar{A}) \subseteq \mathcal{R}(A_1) \oplus \dots \oplus \mathcal{R}(A_\infty) \oplus \mathcal{R}(B)$. Now, $\tilde{\varphi}|_{\mathcal{R}(A)}$ is the desired extension of φ .

Clearly, $\tilde{\varphi}$ is uniquely determined on $\mathcal{R}_0(A)$ with $\tilde{\varphi}(\mathcal{R}_0(A)) \subseteq \mathcal{R}_0(\varphi(A))$. To prove uniqueness on $\mathcal{R}(A)$ and $\tilde{\varphi}(\mathcal{R}(A)) \subseteq \mathcal{R}(\varphi(A))$, it suffices to prove that $\tilde{\varphi}$ is necessarily norm continuous on $\mathcal{R}(A)$. Since A is reversible (Theorem 4.6) we have $a^*a \in A$ for every $a \in \mathcal{R}(A)$; hence it follows from the fact that Jordan homomorphisms of JB-algebras are norm decreasing [7; proof of Lemma 9.3] that

$$\|\tilde{\varphi}(a)\|^2 = \|\tilde{\varphi}(a)^* \tilde{\varphi}(a)\| = \|\tilde{\varphi}(a^*a)\| = \|\varphi(a^*a)\| \leq \|a\|^2,$$

which shows that $\tilde{\varphi}$ is norm continuous.

Finally, assume that φ is 1-1. Then φ is a Jordan isomorphism from A onto $\varphi(A)$. Now $\varphi(A)$ will also have a state space with the 3-ball property, so the above extension proof applies to both φ and φ^{-1} . From this we conclude that $\tilde{\varphi}$ is a real $*$ -isomorphism from $\mathcal{R}(A)$ onto $\mathcal{R}(\varphi(A))$. \square

Remark. An alternative treatment of the I_2 -part in the above proof would be to observe that $\mathcal{R}(A_2)$ is real $*$ -isomorphic to $C(X, M_2(\mathbb{C}))$ and then apply a suitable modification of [35; Th.3.3].

We have already mentioned that a JB-algebra A whose state space K has the 3-ball property, can always be concretely represented as a JC-algebra. For each $\rho \in \partial_e K$ we choose one of the irreducible representations $\pi_\rho: A \rightarrow B(H_\rho)_{sa}$ associated with ρ , then π_ρ is Jordan equivalent with the dense representation $a \mapsto c(\rho) \circ a$ of A into $A_\rho = c(\rho) \circ A^{**}$. Hence it follows from the proof of [7; Lem.9.4] that

$$(5.4) \quad a \mapsto \bigoplus_{\rho \in \partial_e K} \sum \pi_\rho(a)$$

is an isometric Jordan isomorphism of A onto a Jordan subalgebra of $B(H)_{sa}$ where $H = \bigoplus_{\rho \in \partial_e K} H_\rho$. Thus we have realized A as a JC-algebra.

The above construction is non-canonical in that the chosen irreducible representations π_ρ are not unique up to unitary equivalence. This can be amended by choosing instead two mutually conjugate irreducible representations $\pi_\rho: A \rightarrow B(H_\rho)_{sa}$ and $\pi'_\rho: A \rightarrow B(H_\rho)_{sa}$ for each $\rho \in \partial_e K$. Then the map

$$(5.5) \quad a \mapsto \bigoplus_{\rho \in \partial_e K} \Sigma (\pi_\rho(a) \oplus \pi'_\rho(a))$$

is an isometric Jordan isomorphism of A onto a Jordan subalgebra of $B(H)_{sa}$ where $H = \bigoplus_{\rho \in \partial_e K} (H_\rho \oplus H'_\rho)$. Unlike the previous representation (5.4), this "double atomic representation" will have the property that every irreducible representation of A is unitarily equivalent with one of the irreducible representations occurring in (5.5). (Cf. Prop. 2.6, Prop. 3, 7, and Prop. 3.8). However, the most important asset of this representation is that it can be used to define the enveloping C^* -algebra.

Theorem 5.3. Let A be a JB-algebra whose state space K has the 3-ball property. Then there exists a C^* -algebra \mathcal{O} and a Jordan embedding ψ of A into \mathcal{O}_{sa} such that every Jordan homomorphism $\varphi: A \rightarrow B(H)_{sa}$ can be extended uniquely to a $*$ -homomorphism $\tilde{\varphi}$ from \mathcal{O} onto the C^* -algebra generated by $\varphi(A)$ in $B(H)$, and the pair consisting of the C^* -algebra \mathcal{O} and the embedding ψ is in the natural sense unique. Specifically, one can obtain (\mathcal{O}, ψ) by taking ψ to be "the double atomic representation" defined in (5.5) and \mathcal{O} to be the C^* -algebra generated by $\psi(A)$; in fact one shall even have $\mathcal{O} = \mathcal{R}(\psi(A)) + i\mathcal{R}(\psi(A))$ in this case.

Proof. For simplicity of notation we assume that A is given in the representation (4.4), say $A \subseteq B(H_0)_{sa}$. By the definition of conjugacy for GNS-representations we can now express the "double atomic representation" (5.5) as

$\psi: A \rightarrow B(H_0 \oplus H_0)_{sa}$, where

$$(5.6) \quad \psi(a) = a \oplus a^t,$$

where $a \mapsto a^t$ denotes the transpose map with respect to a suitable ortho-normal basis in H_0 . Now ψ will be an isometric Jordan isomorphism of A onto the JC-algebra $\psi(A) \subseteq B(H_0 \oplus H_0)_{sa}$.

Let $x \in \mathcal{R}(\psi(A)) \cap i \mathcal{R}(\psi(A))$ be arbitrary. Since $y \mapsto (y^*)^t$ is a real $*$ -automorphism of $B(H_0)$, it follows that every element of $\mathcal{R}(\psi(A))$ is of the form $b \oplus (b^*)^t$ for some $b \in B(H_0)$. Hence there exist $b, c \in B(H_0)$ such that

$$x = b \oplus (b^*)^t = i(c \oplus (c^*)^t).$$

Thus $b = ic$ and $(b^*)^t = i(c^*)^t$; these force $b = c = 0$, and thus $x = 0$. Thus we have $\mathcal{R}(\psi(A)) \cap i \mathcal{R}(\psi(A)) = \{0\}$. Now we define $\mathcal{O} = \mathcal{R}(\psi(A)) + i \mathcal{R}(\psi(A))$, and we note that it follows from [37; Th.2.1] that \mathcal{O} is a C^* -algebra.

To establish the universal property of \mathcal{O} , we consider a Jordan homomorphism $\varphi: A \rightarrow B(H)_{sa}$. By Theorem 5.2 there exists a real $*$ -homomorphism $\tilde{\varphi}$ from $\mathcal{R}(\psi(A))$ into $B(H)$ which extends φ , in that $\varphi = \tilde{\varphi} \circ \psi$. The elements of \mathcal{O} can be uniquely represented in the form $x + iy$ where $x, y \in \mathcal{R}(\psi(A))$; for such an element we now define

$$\tilde{\varphi}(x + iy) = \tilde{\varphi}(x) + i\tilde{\varphi}(y).$$

It is easily verified that $\tilde{\varphi}$ is a $*$ -homomorphism from \mathcal{O} into $\mathcal{R}(\varphi(A)) + i\mathcal{R}(\varphi(A)) \subseteq C^*(\varphi(A))$ (the C^* -algebra generated by $\varphi(A)$). Since $*$ -homomorphic images of C^* -algebras are C^* -algebras, then $\tilde{\varphi}(\mathcal{O}) = C^*(\varphi(A))$. Uniqueness is evident since A generates \mathcal{O} .

Finally it follows from the results just proved, that the pair (\mathcal{O}, ψ) is unique, i.e. if (\mathcal{O}', ψ') is another pair with the same universal property, then there exists a $*$ -isomorphism Φ

of \mathcal{A} onto \mathcal{A}' such that $\psi' = \Phi \circ \psi$. \square

Remark. The only property of the representation ψ which was used in the above construction, was $\mathcal{R}(\psi(A)) \cap i\mathcal{R}(\psi(A)) = \{0\}$. In fact, for an arbitrary faithful representation of A as a JB-algebra, say $A \subseteq B(H)_{sa}$, then the representation $\varphi: a \mapsto a \oplus a^t$ of A into $B(H \oplus H)_{sa}$ will have this property; thus $\mathcal{R}(\varphi(A)) + i\mathcal{R}(\varphi(A))$ will be a C^* -algebra isomorphic to the enveloping C^* -algebra for A .

We will now study how the state space of the enveloping C^* -algebra is related to the state space of the given JB-algebra.

Lemma 5.4. Let A be a JB-algebra whose state space has the 3-ball property, embedded in its enveloping C^* -algebra \mathcal{A} .
Then the map $\Phi: \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$(5.7) \quad \Phi(x+iy) = x^* + iy^* \quad \text{for } x, y \in \mathcal{R}(A)$$

is a $*$ -anti-automorphism of period two, and the self-adjoint part of the fixed point set of Φ is A .

Proof. By Theorem 5.3, $\mathcal{A} = \mathcal{R}(A) \oplus i\mathcal{R}(A)$, so Φ is well defined. Clearly Φ is a (complex linear) $*$ -anti automorphism of period two. The self-adjoint part of the fixed point set consists of elements $x+iy$ such that $x+iy = x^* + iy^*$ (so $x = x^*$ and $y = y^*$) and such that $(x+iy)^* = x+iy$ (so $y = -y^*$); Thus it is just $\mathcal{R}(A)_{sa} = A$, since A is reversible (Theorem 4.6). \square

Proposition 5.5. Let A be a JB-algebra whose state space \mathcal{K} has the 3-ball property, and let \mathcal{K} be the state space of the enveloping C^* -algebra \mathcal{A} . If $\Phi: \mathcal{A} \rightarrow \mathcal{A}$ is defined as in (5.7), then Φ^* will be an affine automorphism of \mathcal{K} of period two

(a reflection [6 ; § 3]) whose fixed point set \mathcal{K}_0 will be affinely isomorphic to K under the restriction map.

Proof. Clearly Φ^* is an affine automorphism of \mathcal{K} of period two. Clearly also the restriction map sends \mathcal{K} into K , and by the Hahn-Banach theorem this map is onto. Note now that for given $\rho \in \mathcal{K}$ we have $\frac{1}{2}(\Phi^*\rho + \rho) \in \mathcal{K}_0$, and for any $a \in A$:

$$\langle a, \frac{1}{2}(\Phi^*\rho + \rho) \rangle = \frac{1}{2}(\langle \Phi(a), \rho \rangle + \langle a, \rho \rangle) = \langle a, \rho \rangle.$$

Hence ρ and $\frac{1}{2}(\Phi^*\rho + \rho)$ restrict to the same state on M . It follows that the restriction map is an affine homomorphism from \mathcal{K}_0 onto K .

It remains to prove that the restriction map is 1-1 on \mathcal{K}_0 . Consider two states $\rho_1, \rho_2 \in \mathcal{K}_0$ with the same restriction to A . If $x \in \mathcal{R}(A)$ and x is self-adjoint, then $x \in \mathcal{R}(A)_{sa} = A$ since A is reversible; hence $\langle x, \rho_1 \rangle = \langle x, \rho_2 \rangle$. If $x \in \mathcal{R}(A)$ and x is skew, then for $j = 1, 2$:

$$\langle x, \rho_j \rangle = \langle x, \Phi^*\rho_j \rangle = \langle x^*, \rho_j \rangle = -\langle x, \rho_j \rangle,$$

so $\langle x, \rho_1 \rangle = \langle x, \rho_2 \rangle = 0$. Applying these results to the self-adjoint and the skew component of an arbitrary $x \in \mathcal{R}(A)$, we conclude that $\langle x, \rho_1 \rangle = \langle x, \rho_2 \rangle$. Hence $\langle x, \rho_1 \rangle = \langle x, \rho_2 \rangle$ for all $x \in \mathcal{O} = \mathcal{R}(A) \oplus i\mathcal{R}(A)$ as well. This proves the injectivity of the restriction map from \mathcal{K}_0 to K . \square

Proposition 5.6. Let A be a JB-algebra whose state space K has the 3-ball property, and let \mathcal{K} be the state space of the enveloping C^* -algebra \mathcal{O} . Then the restriction map sends $\partial_0 \mathcal{K}$ onto $\partial_0 K$. Moreover, if $\rho \in \partial_0 \mathcal{K}$ and $\sigma = \Phi^*\rho$, then ρ and σ will restrict to the same pure state $\tau \in \partial_0 K$,

and the two GNS-representations π_ρ and π_σ of \mathcal{O} associated with ρ and σ will restrict to conjugate irreducible representations of A associated with τ .

Proof. It is easily verified that every $\tau \in \partial_e K$ is the restriction of some $\rho \in \partial_e \mathcal{K}$; in fact the (non-empty) set of states in \mathcal{K} which restrict to τ is a w^* -closed face, which must contain points from $\partial_e \mathcal{K}$ by the Krein-Milman theorem.

We will next prove that $\partial_e \mathcal{K}$ is mapped into $\partial_e K$. Let $\rho \in \partial_e \mathcal{K}$ be arbitrary, and let π_ρ be the associated GNS-representation of \mathcal{O} . Since A generates \mathcal{O} as a C^* -algebra, it follows that $\pi_\rho(A)$ will generate $\pi_\rho(\mathcal{O})$. Then it follows from the irreducibility of $\pi_\rho(\mathcal{O})$ that $\pi_\rho(A)$ is an irreducible JC-algebra. Now by Proposition 3.12 the map $a \mapsto (\pi_\rho(a)\xi|\xi)$ is a pure state on A where ξ is the cyclic vector for the GNS-representation π_ρ of \mathcal{O} . But this state is nothing else than $\tau = \rho|_A$, and $\pi_\rho|_A$ will be an irreducible representation of A corresponding to τ . Since the elements of A are fixed under Φ , we have for each $a \in A$:

$$\langle a, \sigma \rangle = \langle a, \Phi^* \rho \rangle = \langle \Phi a, \rho \rangle = \langle a, \sigma \rangle .$$

Hence ρ and σ restrict to the same pure state τ on A .

It remains to prove that the irreducible representations $\pi_\rho|_A$ and $\pi_\sigma|_A$ of A are conjugate. In this connection we note that for $a, b \in A$ we have $\Phi([a, b]) = -[a, b]$, since $[a, b] \in \mathcal{R}(A)$ and $[a, b]^* = -[a, b]$. Denoting the distinguished cyclic vector for π_σ by η , we now find

$$\begin{aligned} ([\pi_\sigma(a), \pi_\sigma(b)]\eta|\eta) &= (\pi_\sigma([a, b])\eta|\eta) = \langle [a, b], \sigma \rangle = \langle \Phi([a, b]), \rho \rangle \\ &= - \langle [a, b], \rho \rangle = - (\pi_\rho([a, b])\xi|\xi) = - ([\pi_\rho(a), \pi_\rho(b)]\xi|\xi) . \end{aligned}$$

By Proposition 3.11, this completes the proof. \square

Corollary 5.7. Let the assumptions be as in Proposition 4.6 and consider $\rho \in \partial_e \mathcal{K}$. Then $\rho \in \mathcal{K}_0$ iff the GNS-representation π_ρ of \mathcal{O} which is associated with ρ is 1-dimensional.

Proof. Assume first that π_ρ is 1-dimensional. Then $\pi_\rho(a) = \rho(a)I$ (where I is the identity operator), so ρ is a multiplicative linear functional on \mathcal{O} . Clearly the same applies to $\sigma = \mathfrak{F}^*\rho$. By Proposition 5.6, ρ and σ always coincide on A , and by multiplicativity they will coincide on $\mathcal{R}(A)$, and then on $\mathcal{O} = \mathcal{R}(A) \oplus i\mathcal{R}(A)$. Thus $\rho = \mathfrak{F}^*\rho$, so $\rho \in \mathcal{K}_0$.

Assume next that $\rho \in \mathcal{K}_0$, i.e. $\mathfrak{F}^*\rho = \rho$. Then the two mutually conjugate irreducible representations of A which are associated to $\rho|_A$, must coincide by virtue of Proposition 4.6. But this is possible only if the dimension is one (cf. Proposition 3.9). In this case also $\dim \pi_\rho = 1$ by Proposition 5.6. \square

We will now show that the number of pure states on \mathcal{O} which restrict to a given pure state on A is either two or one; the last case will occur iff the irreducible representations associated with the given state is of dimension one (i.e. if this state is a split face, cf. § 2).

Lemma 5.8. Let A be a JB-algebra whose state space K has the 3-ball property, and let \mathcal{O} be the enveloping C^* -algebra with state space \mathcal{K} . If $\rho \in \partial_e \mathcal{K}$, then either $\mathfrak{F}^*(F_\rho) \cap F_\rho = \emptyset$ or $\mathfrak{F}^*(F_\rho) = F_\rho$; the latter will occur iff $F_\rho = \{\rho\}$.

Proof. Since F_ρ and $\mathfrak{F}^*(F_\rho) = F_{\mathfrak{F}^*(\rho)}$ are minimal split faces, they clearly must be either disjoint or equal. We have to

show that they are equal iff $F_\rho = \{\rho\}$.

Assume first $F = \{\rho\}$. Then $\dim \pi_\rho = 1$ (where π_ρ denotes the associated GNS-representation of \mathcal{O} as before); by Corollary 5.7, $\rho \in \mathcal{K}_0$. Hence $\Phi^*\rho = \rho$, and so $\Phi^*(F_\rho) = F_{\Phi^*\rho} = F_\rho$.

Conversely, we assume $F_\rho = \Phi^*(F_\rho) = F_{\Phi^*(\rho)}$. By known results on C^* -algebras, π_ρ and $\pi_{\Phi^*\rho}$ are unitarily equivalent (These are the results we have generalized to JB-algebras in Proposition 2.6. A reference for the C^* -algebra case is [16; §5.3]). It follows that π_ρ and $\pi_{\Phi^*\rho}$ will restrict to unitarily equivalent representations of A . But by Proposition 5.6, π_ρ and $\pi_{\Phi^*\rho}$ will restrict to mutually conjugate irreducible representations associated with the common restriction of ρ and $\Phi^*(\rho)$. Hence we must have $\dim \pi_\rho = \dim (\pi_\rho|_A) = 1$ (cf. Proposition 3.9), and so $F_\rho = \{\rho\}$. \square

Proposition 5.9. Let A be a JB-algebra whose state space K has the 3-ball property, and let \mathcal{O} be the enveloping C^* -algebra with state space \mathcal{K} . If $\rho, \sigma \in \partial_{\mathcal{K}}$ and $\rho|_A = \sigma|_A$, then either $\sigma = \rho$ or $\sigma = \Phi^*\rho$.

Proof. Note that if $x \in \mathcal{O}_{sa}$, then $(I+\Phi)x$ is fixed by Φ , and so it lies in A (Lemma 5.4). Thus

$$\langle x, (I+\Phi)^*\rho \rangle = \langle (I+\Phi)x, \rho \rangle = \langle (I+\Phi)x, \sigma \rangle = \langle x, (I+\Phi)^*\sigma \rangle,$$

which implies

$$(5.8) \quad \frac{1}{2}(\rho + \Phi^*\rho) = \frac{1}{2}(\sigma + \Phi^*\sigma).$$

Thus $\frac{1}{2}(\sigma + \Phi^*\sigma) \in \text{co}(F_\rho \cup \Phi^*(F_\rho))$. Since the latter set is a face of \mathcal{K} (cf. e.g. [1; Cor.II.6.8]), it must contain σ . Since σ is an extreme point, it must lie in either F_ρ or $\Phi^*(F_\rho)$.

Now the conclusion is trivial if $F_\rho = \{\rho\}$. If $F_\rho \neq \{\rho\}$, then it follows from Lemma 5.8 that F_ρ and $\Phi^*(F_\rho)$ are disjoint. Hence $\Phi^*(F_\rho) \subset F'_\rho$ (where the "prime" denotes the complementary split face, as usual). By the uniqueness statement involved in the definition of a split face, it follows from (5.8) that $\sigma = \rho$ in case $\sigma \in F_\rho$, and $\sigma = \Phi^*\rho$ in case $\sigma \in \Phi^*F_\rho$. The proof is complete. \square

In the corollary below we will use the notation introduced in (2.2) for the state space K of A , and also the similar notions for the state space \mathcal{K} of the enveloping C^* -algebra \mathcal{O} . Then we can state:

Corollary 5.10. Let the assumptions be as in Proposition 5.9.
Then the restriction map $r: \mathcal{K} \rightarrow K$ will be one to one from $\delta_{e,0}\mathcal{K}$
onto $\delta_{e,0}K$ and two to one from $\delta_{e,1}\mathcal{K}$ onto $\delta_{e,1}K$.

Proof. By Proposition 5.6, $r(\delta_e \mathcal{K}) = \delta_e K$.

If $\rho \in \delta_{e,0}\mathcal{K}$, then it follows from Corollary 5.7 that $\rho \in \mathcal{K}_0$, i.e. $\Phi^*\rho = \rho$. By Proposition 5.9, ρ is the only element in $\delta_e \mathcal{K}$ which restricts to the state $r(\rho)$ on A .

If $\rho \in \delta_{e,1}\mathcal{K}$, then $\rho \notin \mathcal{K}_0$, i.e. $\Phi^*\rho \neq \rho$. Now it follows from Proposition 5.9 that the elements of $\delta_e \mathcal{K}$ which restrict to the state $r(\rho)$ on A , are precisely ρ and $\Phi^*\rho$. \square

We will now very briefly indicate how the enveloping C^* -algebra can be used to approach our main problem. If A is a JB-algebra whose state space K has the 3-ball property, and if there exists a faithful representation ψ of A onto the self-adjoint part of a C^* -algebra \mathcal{B} , then ψ can be extended to a $*$ -homomorphism $\tilde{\psi}$ of the enveloping C^* -algebra \mathcal{O} onto \mathcal{B} ,

so

$$(\mathcal{O}/\ker \tilde{\Psi})_{sa} \cong \mathcal{O}_{sa} \cong A .$$

Thus, A will be (Jordan isomorphic to) the self-adjoint part of a C^* -algebra iff \mathcal{O} contains a closed two-sided ideal J such that $(\mathcal{O}/J)_{sa} \cong A$. Dualizing, one can rephrase this statement in a form which involves \mathcal{K} and K rather than \mathcal{O} and A . However, before going any further into the geometry of the state spaces, we will show by an example that the 3-ball property alone does not suffice to guarantee that A is the self-adjoint part of a C^* -algebra, so an additional geometric condition is needed on K .

We now define $\mathcal{O} = C(T, M_2(\mathbb{C}))$ and

$$(5.9) \quad A = \{f \in \mathcal{O} \mid f = f^*, f(-\lambda) = f(\lambda)^t \text{ all } \lambda \in T\} ,$$

where T is the unit circle, $\lambda \mapsto -\lambda$ the antipodal map, and $m \mapsto m^t$ the transpose map. Then A is a JB-algebra, in fact a Jordan subalgebra of \mathcal{O}_{sa} . Note that the condition $f(-\lambda) = f(\lambda)^t$ of (5.9) is equivalent to $f(-\lambda) = [f(\lambda)^*]^t$ since f is already supposed to be self-adjoint. Thus A is contained in the norm-closed real $*$ -algebra

$$\mathcal{R} = \{f \in \mathcal{O} \mid f(-\lambda) = [f(\lambda)^*]^t \text{ all } \lambda \in T\} .$$

Clearly $\mathcal{R}_{sa} = A$, and we also observe that $\mathcal{R} = M_2(B)$ where B is the norm-closed real $*$ -algebra of all $g \in C(T, \mathbb{C})$ such that $g(-\lambda) = g(\lambda)^-$ for all $\lambda \in T$. Note that \mathcal{R} is generated as a real algebra by \mathcal{R}_{sa} , i.e. $\mathcal{R} = \mathcal{R}_0(A) = \mathcal{R}(A)$. Note also that every $g \in C(T, \mathbb{C})$ can be uniquely decomposed as $g = g_1 + i g_2$ with $g_1, g_2 \in B$, in fact $g_1(\lambda) = \frac{1}{2}(g(\lambda) + g(-\lambda)^-)$ and $g_2(\lambda) = \frac{1}{2i}(g(\lambda) - g(-\lambda)^-)$. From this it follows that $\mathcal{O} = \mathcal{R} \oplus i\mathcal{R}$. We will see below that the state space of A has the 3-ball property,

so A has an enveloping C^* -algebra. In fact, as mentioned in the remark after Theorem 5.3 $\mathcal{O}_1 = \mathbb{R} \oplus i\mathbb{R}$ will be the enveloping C^* -algebra for A .

Proposition 5.11. The state space K of the JB-algebra A defined in (5.9) has the 3-ball property, but A is not isomorphic to the self-adjoint part of any C^* -algebra. However, A^{**} is isomorphic to the self-adjoint part of a von Neumann algebra, and there is an affine (w^* -discontinuous) isomorphism from K onto the state space of a C^* -algebra.

Proof. 1.) It is known (and easily verified) that each pure state on $\mathcal{O}_1 = C(T, M_2(\mathbb{C}))$ is of the form $f \mapsto \langle f(\lambda), \alpha \rangle$ for some $\lambda \in T$ and some pure state α on $M_2(\mathbb{C})$, and that the associated GNS-representation is $f \mapsto f(\lambda)$ (up to unitary equivalence). By Proposition 4.6, every pure state ρ on A is of the form

$$(5.10) \quad \rho: f \mapsto \langle f(\lambda_\rho), \alpha_\rho \rangle$$

where $\lambda_\rho \in T$ and α_ρ is a pure state on $M_2(\mathbb{C})_{sa}$, and it has an associated irreducible representation of the form

$$(5.11) \quad \pi_\rho: f \mapsto f(\lambda_\rho).$$

Note that in each case $\pi_\rho(A) = M_2(\mathbb{C})_{sa}$, and so by Corollary 3.6, K has the 3-ball property.

2.) Next, suppose for contradiction that $A \cong \mathcal{B}_{sa}$ where \mathcal{B} is a C^* -algebra, say $\psi: A \rightarrow \mathcal{B}_{sa}$ is the isomorphism. Then for every $\lambda \in T$, the map $\varphi_\lambda: b \mapsto \psi^{-1}(b)(\lambda)$ is a Joran homomorphism of \mathcal{B}_{sa} onto $M_2(\mathbb{C})_{sa}$. Let $\tilde{\varphi}_\lambda: \mathcal{B} \rightarrow M_2(\mathbb{C})$ be the complex linear extension of φ_λ . By a known result [35; Th.3.3] $\tilde{\varphi}_\lambda$ is either a $*$ -homomorphism or a $*$ -anti-homomorphism for each $\lambda \in T$.

We will show that one of these alternatives for $\tilde{\varphi}_\lambda$ must prevail for all $\lambda \in T$ by continuity, and then arrive at a contradiction by showing that the "twisting condition" $f(-\lambda) = f(\lambda)^t$ in the definition of A forces opposite alternatives for $\tilde{\varphi}$ at λ and $-\lambda$. The details follow.

For arbitrary $f, g \in A$, we have $i[\psi(f), \psi(g)] \in \mathcal{B}_{sa}$, so $h = \psi^{-1}(i[\psi(f), \psi(g)])$ exists in A . Thus, for each $\lambda \in T$ we define:

$$(5.12) \quad h(\lambda) = i\tilde{\varphi}_\lambda([\psi(f), \psi(g)]) = \pm i[f(\lambda), g(\lambda)],$$

where the sign depends on whether $\tilde{\varphi}_\lambda$ is a $*$ -homomorphism or a $*$ -anti-homomorphism.

Now let $s_1, s_2, s_3 \in M_2(\mathbb{C})_{sa}$ be the elementary spin matrices (cf. (4.4)), and note that s_1, s_2 are symmetric, while s_3 is antisymmetric. Also let f and g be the constant functions $f(\lambda) \equiv s_1$, $g(\lambda) \equiv s_2$. Then $f, g \in A$, and the corresponding function $h \in A$ must satisfy

$$(5.13) \quad h(\lambda) = \pm i[s_1, s_2] = \pm 2s_3, \quad \text{all } \lambda \in T.$$

Since h is continuous, the same sign must hold throughout in (5.13), say $h(\lambda) = +2s_3$ for all $\lambda \in T$. But now $h(-\lambda) = h(\lambda)^t = -2s_3$, a contradiction.

3.) It remains to prove that A^{**} can be embedded as the self-adjoint part of a von Neumann algebra. This result (which we shall not need in the sequel) can be proved in various ways, and we will give a proof which has the advantage of actually producing a C^* -algebra \mathcal{A}_0 such that $A^{**} \cong (\mathcal{A}_0^{**})_{sa}$.

Since \mathcal{A}_{sa} consists of all self-adjoint 2×2 -matrices over the Banach space $C(T, \mathbb{C})$, then \mathcal{A}_{sa}^* can be represented by all self-adjoint 2×2 -matrices over the Banach space of (regular Borel)

measures on T . Thus we may view the $M_2(\mathbb{C})$ -valued bounded Borel functions as embedded in \mathcal{O}^{**} . Let E be the semicircle $\{\lambda = e^{i\varphi} \mid 0 \leq \varphi \leq \pi\}$ and let $F \subseteq \mathcal{K}$ be defined by

$$F = \{\rho \in \mathcal{K} \mid \langle \chi_E, \rho \rangle = 1\}.$$

We will show that the restriction map sends F bijectively onto K . We begin by defining $\psi: \mathcal{O}_{sa} \rightarrow \bar{A}$ by

$$\psi(f)(\lambda) = \chi_E(\lambda)f(\lambda) + \chi_{T \setminus E}f(-\lambda)^t.$$

(Note, $\psi(\mathcal{O}_{sa}) \subseteq \bar{A}$ follows from the dominated convergence theorem). Now if $\rho_1, \rho_2 \in F$ agree on A , then they agree on \bar{A} , so if we define $f'(\lambda) = f(-\lambda)$ for $\lambda \in T$, then for all $f \in \mathcal{O}$:

$$\begin{aligned} \langle f, \rho_1 \rangle &= \langle \chi_E f, \rho_1 \rangle = \langle \chi_E f + \chi_{T \setminus E}(f')^t, \rho_1 \rangle \\ &= \langle \chi_E f + \chi_{T \setminus E}(f')^t, \rho_2 \rangle = \langle f, \rho_2 \rangle. \end{aligned}$$

Thus the restriction map is injective. To show it is surjective, for $\sigma \in K$ let $\tilde{\sigma} \in \mathcal{K}$ be any extension and observe that the state on \mathcal{O} given by

$$f \mapsto \langle \chi_E f + \chi_{T \setminus E}(f')^t, \tilde{\sigma} \rangle$$

is in F and restricts to σ .

But in a similar way F is affinely isomorphic to the state space of A_0 , where

$$A_0 = \{f \in \mathcal{O} \mid f = f^*, f(-\lambda) = f(\lambda) \text{ all } \lambda \in T\}.$$

Thus A and A_0 have affinely isomorphic state spaces, and so $A^{**} \cong A_0^{**}$. But A_0 is the self-adjoint part of the C^* -algebra $\mathcal{O}_0 = \{f \in \mathcal{O} \mid f(\lambda) = f(-\lambda)\}$; hence A^{**} is the self-adjoint part of the von Neumann algebra \mathcal{O}_0^{**} . \square

§ 6. Orientations.

Throughout this section we assume that A is a JB-algebra whose state space K has the 3-ball property. We will begin by studying the notion of orientation for the 3-balls $B(\rho, \sigma) \subseteq K$ and its relationship with irreducible representations of A .

By definition, for every 3-ball B there exists a parametrization, i.e. an affine isomorphism ψ from B onto the unit ball of \mathbb{R}^3 . Now an orientation of B can be defined to be an equivalence class of such parametrizations, with ψ_1 being equivalent to ψ_2 if (the orthogonal) transformation $\psi_2 \circ \psi_1^{-1}$ has determinant $+1$.

One can also view an orientation as a choice of a vector product. If \times denotes the usual vector product in \mathbb{R}^3 , then for each parametrization ψ of B one has an induced product:

$$(6.1) \quad \omega \times_{\psi} \tau = \psi^{-1}(\psi(\omega) \times \psi(\tau)) .$$

Two parametrizations ψ and φ will be equivalent iff $\times_{\psi} = \times_{\varphi}$. By convenient abuse of terminology, we will use the term "an orientation of B " also to mean one of the two possible products on B which arise in this way. Note that if $(\omega, \tau) \rightarrow \omega \times \tau$ is one orientation of B , then the other ("opposite") orientation will be $(\omega, \tau) \rightarrow \tau \times \omega$.

Recall from [5; Prop.1.13] that there is a 1-1 correspondence between minimal idempotents (atoms) $p \in A^{**}$ and pure states $\rho \in \partial_e K$, given by $\langle p, \rho \rangle = 1$. As in [6] we write $\rho = \hat{p}$, and for convenience we now also write $p = \check{\rho}$. In [6; Lem.5.5] it was shown that the map $\rho \rightarrow \check{\rho}$ extends to an isomorphism of the linear span of $\partial_e K$ in A^* onto the linear span A_f of all minimal idempotents

in A^{**} . For convenience we will also use the notation $\rho \mapsto \hat{\rho}$ (and $p \rightarrow \hat{p}$) for the extended map from $\text{lin } \partial_e K$ onto A_f (and from A_f to $\text{lin } \partial_e K$). Finally, if $\rho, \sigma \in \partial_e K$ are not separated by a split face, then we will denote the center of the ball $B(\rho, \sigma)$ by $\gamma(\rho, \sigma)$.

Lemma 6.1. Let ρ, σ be distinct extreme points of K not separated by a split face, and let $r = \rho, s = \sigma$. Then

$$(6.2) \quad \gamma(\rho, \sigma) = \frac{1}{2} \|(r-s)^2\|^{-1} [(r-s)^2]^\wedge = \frac{1}{2} (r \vee s)^\wedge.$$

Proof. Note first that $\text{tr}(r-s) = 0$ where "tr" denotes the trace in $\{(r \vee s)A^{**}(r \vee s)\} \cong M_2(\mathbb{C})_{sa}$. Hence $r-s = \alpha(p-q)$ where p and q are orthogonal minimal idempotents in $\{(r \vee s)A^{**}(r \vee s)\}$, and so $(r-s)^2 = \alpha^2(p+q)$. Since $p+q$ is an idempotent, it has norm one in A^{**} , so $\alpha^2 = \|(r-s)^2\|$. Hence

$$(6.3) \quad p+q = \|(r-s)^2\|^{-1} (r-s)^2$$

Recall now that $B(\rho, \sigma)$ is the normal state space of $\{(r \vee s)A^{**}(r \vee s)\}$, so the orthogonal minimal idempotents p, q will determine antipodal extreme points \hat{p}, \hat{q} of $B(\rho, \sigma)$. Thus

$$\gamma(\rho, \sigma) = \frac{1}{2}(\hat{p} + \hat{q}) = \frac{1}{2} \|(r-s)^2\|^{-1} [(r-s)^2]^\wedge.$$

This is the first equality of (6.2). By (6.3) we also get

$$\frac{1}{2} \|(r-s)^2\|^{-1} [(r-s)^2]^\wedge = \frac{1}{2} (p+q)^\wedge = \frac{1}{2} (p \vee q)^\wedge = \frac{1}{2} (r \vee s)^\wedge.$$

As before, we denote the σ -weakly continuous extension to A^{**} of an irreducible representation $\pi : A \rightarrow B(H)_{sa}$ by $\tilde{\pi}$, and we recall that if π is associated with a pure state ρ then $\tilde{\pi}$ is an isomorphism of $A_\rho = c(\rho) \circ A^{**}$ onto $B(H)_{sa}$. \square

Proposition 6.2. Let $\rho, \sigma \in K$ be distinct extreme points not separated by a split face, and let $\pi: A \rightarrow B(H)_{sa}$ be an irreducible representation associated with ρ (and σ). Then there is an orientation of $B(\rho, \sigma)$ such that for all pairs $\omega, \tau \in B(\rho, \sigma)$:

$$(6.4) \quad \omega \times \tau = \gamma(\omega, \tau) + \{(\tilde{\pi}|_{A_\rho})^{-1}(i[\tilde{\pi}(\check{\omega}), \tilde{\pi}(\check{\tau})])\}^\wedge.$$

This orientation will remain unchanged if π is replaced by a unitarily equivalent representation, but it will be reversed if π is replaced by a conjugate representation.

Proof. Let $r = \check{\rho}$ and $s = \check{\sigma}$. Then the image of $B(\rho, \sigma)$ under the map $\omega \mapsto \check{\omega}$ will be the positive elements of trace one in $\{(rvs)A^{**}(rvs)\} = \{(rvs)A_\rho(rvs)\}$. Let $p = \tilde{\pi}(rvs) \in B(H)$, then p is a two-dimensional projection, and $\tilde{\pi}$ restricts to an isomorphism of $\{(rvs)A_\rho(rvs)\}$ onto $pB(H)_{sa}p$. If we choose an orthonormal basis for H , we can represent the positive elements of trace one in $pB(H)_{sa}p$ by ("density") matrices of the form

$$(6.5) \quad D = \frac{1}{2} \begin{pmatrix} 1 + \lambda_1 & \lambda_2 + i\lambda_3 \\ \lambda_2 - i\lambda_3 & 1 - \lambda_1 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ are real and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1$.

If the ordinary vector product in \mathbb{R}^3 is pulled back to the density matrices by the map $D \mapsto (\lambda_1, \lambda_2, \lambda_3)$, the resulting product will be $(D_1, D_2) \mapsto \frac{1}{2}I + i[D_1, D_2]$, as may be verified by direct calculation. Passing to $pB(H)_{sa}p$ and then pulling back to $\{(rvs)A_\rho(rvs)\}$ by the (restriction of) the isomorphism $\tilde{\pi}|_{A_\rho}$, we get the following product where $x, y \in \{(rvs)A_\rho(rvs)\}$:

$$(x, y) \mapsto \frac{1}{2}(rvs) + (\tilde{\pi}|_{A_\rho})^{-1}(i[\tilde{\pi}(x), \tilde{\pi}(y)])$$

(Recall that the identity element in $pB(H)p$ is $p = \tilde{\pi}(rvs)$).

Finally, we pull back to $B(\rho, \sigma)$ by the map $\omega \mapsto \omega$ and obtain the following product in $B(\rho, \sigma)$:

$$(\omega, \tau) \mapsto \frac{1}{2}(rVs)^{\wedge} + \{(\tilde{\pi}|_{A_{\rho}})^{-1}(i[\tilde{\pi}(\tilde{\omega}), \tilde{\pi}(\tilde{\tau})])\}^{\wedge}.$$

By Lemma 6.1, $\frac{1}{2}(rVs)^{\wedge} = \gamma(\rho, \sigma)$, so we have an orientation of $B(\rho, \sigma)$ satisfying (6.4).

The last statement of the proposition follows from the basic properties of unitarily equivalent and conjugate representations, established in § 3. \square

If π is an irreducible representation associated with $\rho \in \partial_e K$, then for every $\sigma \in F_{\rho}$, $\sigma \neq \rho$ we will say that the orientation described in Proposition 6.2 is induced on $B(\rho, \sigma)$ by π .

We now turn to the study of simultaneous orientations of sets of balls $B(\rho, \sigma)$ with $\rho, \sigma \in \partial_e K$. By the above result, there are two natural (and mutually opposite) ways in which the balls within one F_{ρ} ($\rho \in \partial_e K$) can be oriented: one where all the orientations are induced by an irreducible representation π associated with ρ , another where all the orientations are induced by a representation conjugate to π .

We will now proceed to define a notion of consistency of orientations for a set of balls, not all contained in a common F_{ρ} with $\rho \in \partial_e K$. Intuitively, the idea is to avoid any sudden reversal of orientation by passage to "neighbouring" balls. Here the term "neighbouring" relates to the w^* -topology on K . However, the w^* -topology is defined for single states, not for balls. To specify a ball $B(\rho, \sigma)$ one needs two states, and it is not appropriate to define proximity for the balls $B(\rho, \sigma)$ by proximity of the generating pairs $(\rho, \sigma) \in \partial_e K \times \partial_e K$. Even in the case of a C^* -algebra there may exist such pairs which are close to each other, while the

corresponding balls are far apart. More specifically, the map $(\rho, \sigma) \mapsto \gamma(\rho, \sigma)$ will not be w^* -continuous for C^* -algebras in general. In fact, one can even have discontinuity of the map $\sigma \mapsto \gamma(\rho, \sigma)$ where ρ is a fixed pure state and σ is a pure state in the split face F_ρ generated by ρ . We will sketch one example to this effect where K is the state space of a UHF-algebra \mathcal{O} of type $\{2^n\}$ (i.e. a direct limit of $2^n \times 2^n$ -matrix algebras, cf. [20]). (Since we shall not need this example later on, we will omit the details of the verification). By an inductive argument one can construct $\sigma_0, \sigma_1, \sigma_2, \dots$ in $\partial_e K$ such that $\sigma_n \in F_{\sigma_0}$ for all $n = 0, 1, 2, \dots$, such that σ_n and σ_m are antipodal in $B(\rho_n, \rho_m)$ when $n \neq m$, and such that σ_n converges to σ_0 in the w^* -topology. Consider now the GNS-representation $\pi: \mathcal{O} \rightarrow B(H)$ associated with ρ_0 , and let the pure states $\rho_0, \rho_1, \rho_2, \dots$ be represented by the unit vectors $\xi_0, \xi_1, \xi_2, \dots$ in H . Also define $\eta = 2^{-\frac{1}{2}} \xi_0 + \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} \xi_n$, and let ρ be the state represented by this vector. By evaluating the occurring states at an appropriate $a \in A$, one can now show that $\gamma(\rho, \sigma_n)$ does not converge to $\gamma(\rho, \sigma_0)$. Hence, $\sigma \mapsto \gamma(\rho, \sigma)$ is w^* -discontinuous.

In the counterexample above it was essential that σ was a free variable independent of ρ (subject only to the requirement $\sigma \in F_\rho$). We will now see that the situation is different if σ is taken to be an appropriate w^* -continuous function of ρ . Our approach will be based on the key observation that there is associated to each $a \in A$ a natural w^* -continuous transformation of pure states, $\rho \mapsto \rho_a$, which maps each minimal split face F_ρ into itself and for which $\rho \mapsto \gamma(\rho, \rho_a)$ is w^* -continuous (whenever defined). Now it suffices to vary ρ continuously in the w^* -topology of $\partial_e K$ and keep $a \in A$

fixed, then the ball $B(\rho, \rho_a)$ will "pass continuously to neighbouring balls". The details follow.

Definition. For given $a \in A$ and each $\rho \in \partial_e K$ such that $\langle a^2, \rho \rangle \neq 0$, the transformed state ρ_a is given by

$$(6.6) \quad \langle b, \rho_a \rangle = \langle a^2, \rho \rangle^{-1} \langle \{aba\}, \rho \rangle \quad \text{all } b \in A.$$

Suppose now that $\rho \in \partial_e K$ and that $\pi: A \rightarrow B(H)_{sa}$ is an irreducible representation associated with ρ and that $\xi_\rho \in H$ represents ρ (with respect to π). If $a \in A$ and $\langle a^2, \rho \rangle \neq 0$, then for all $b \in A$:

$$\langle b, \rho_a \rangle = (\pi(a)\xi_\rho | \pi(a)\xi_\rho)^{-1} (\pi(b)\pi(a)\xi_\rho | \pi(a)\xi_\rho),$$

and hence the transformed state ρ_a is represented by the vector

$$(6.7) \quad \xi_{\rho_a} = \|\pi(a)\xi_\rho\|^{-1} \pi(a)\xi_\rho.$$

Thus, in the representation π the transformation of states by a simply means to apply the operator $\pi(a)$ to a vector representing the state.

By formula (6.7) and Proposition 3.12, ρ_a is a pure state contained in F_ρ . Clearly also, $\rho \mapsto \rho_a$ is a w^* -continuous function from its domain

$$(6.8) \quad V_a = \{\rho \in \partial_e K | \langle a^2, \rho \rangle \neq 0\}$$

into $\partial_e K$ for every $a \in A$.

Note also that the vector ξ_{ρ_a} will represent the same state as ξ_ρ (i.e. ρ itself) iff $|(\xi_{\rho_a} | \xi_\rho)| = 1$; by (6.7) this is equivalent to $(\pi(a)\xi_\rho | \xi_\rho) = \|\pi(a)\xi_\rho\|$, and this in turn is equivalent to $\langle a, \rho \rangle = \langle a^2, \rho \rangle^{\frac{1}{2}}$. Hence $\rho_a = \rho$ iff $\langle a, \rho \rangle^2 = \langle a^2, \rho \rangle$. Thus

ρ and ρ_a will determine a (proper) 3-ball $B(\rho, \rho_a)$ iff $\langle a, \rho \rangle^2 < \langle a^2, \rho \rangle$. (Note that $\langle a, \rho \rangle^2 \leq \langle a^2, \rho \rangle$ always holds by the Schwartz' inequality). Hence the map $\rho \mapsto \gamma(\rho, \rho_a)$ is defined on the set

$$(6.9) \quad W_a = \{\rho \in \partial_e K \mid \langle a, \rho \rangle^2 < \langle a^2, \rho \rangle\} \subseteq V_a.$$

We will now show that this map is w^* -continuous from W_a into K for every $a \in A$.

Lemma 6.3. Let $\pi : A \rightarrow B(H)_{sa}$ be an irreducible representation associated with $\rho \in \partial_e K$. For given $\omega \in \partial_e K$ the operator $\tilde{\pi}(\tilde{\omega})$ is non-zero iff $\omega \in F_\rho$; in this case $\tilde{\pi}(\tilde{\omega})$ is the projection onto $\text{lin}(\xi_\omega)$ where $\xi_\omega \in H$ is a vector representing ω . More generally, for $\omega \in \text{lin } \partial_e K$ the operator $\tilde{\pi}(\tilde{\omega})$ is of finite rank and

$$(6.10) \quad \langle b, \omega \rangle = \text{tr}(\tilde{\pi}(\tilde{\omega})\pi(b)) \quad \text{all } b \in A.$$

Proof. For every $\omega \in \partial_e K$, $\tilde{\omega}$ is a minimal idempotent in A^{**} , so $\tilde{\pi}(\tilde{\omega})$ is either a one-dimensional projection or zero.

If $\omega \notin F_\rho$, then $\rho \in F'_\rho$ so $\langle c(\rho), \omega \rangle = 0$. This implies that $c(\rho)$ and $\tilde{\omega}$ are orthogonal (since clearly $\tilde{\omega} \not\leq c(\rho)$). Now, $\tilde{\pi}(c(\rho)) = 1$ implies $\tilde{\pi}(\tilde{\omega}) = 0$.

If $\omega \in F_\rho$, then it follows from Proposition 3.8 that there exists a vector $\xi_\omega \in H$ which represents ω . Hence

$$1 = \langle \tilde{\omega}, \omega \rangle = (\tilde{\pi}(\omega)\xi_\omega \mid \xi_\omega),$$

so the one-dimensional projection $\tilde{\pi}(\omega)$ must be onto $\text{lin}(\xi_\omega)$.

Now the last statement of the lemma follows by linearity. \square

The next lemma gives an alternative (dual) definition of ρ_a where the map $U_a : b \mapsto \{aba\}$ is applied to $\check{\rho}$ rather than U_a^* to ρ , as in (6.6).

Lemma 6.4. If $a \in A$, $\rho \in \partial_e K$, and $\langle a^2, \rho \rangle \neq 0$, then

$$(6.11) \quad \rho_a = \langle a^2, \rho \rangle^{-1} \{a\check{a}a\}^\wedge.$$

Proof. Let π be an irreducible representation associated with ρ . By Lemma 6.3 and the commutativity property of the trace, for each $b \in A$:

$$\begin{aligned} \langle \{aba\}, \rho \rangle &= \text{tr}(\tilde{\pi}(\check{\rho})\pi(\{aba\})) \\ &= \text{tr}(\tilde{\pi}(\check{\rho})\pi(a)\pi(b)\pi(a)) \\ &= \text{tr}(\pi(a)\tilde{\pi}(\check{\rho})\pi(a)\pi(b)) \\ &= \text{tr}(\tilde{\pi}(\{a\check{a}a\})\pi(b)). \end{aligned}$$

Note that since $\tilde{\pi}(\check{\rho})$ is a one-dimensional projection, then $\tilde{\pi}(\{a\check{a}a\}) = \pi(a)\tilde{\pi}(\check{\rho})\pi(a)$ will be a scalar multiple of a one-dimensional projection. Hence $\{a\check{a}a\} \in A_f$, so $\{a\check{a}a\}^\wedge$ is defined. Now by Lemma 6.3

$$\langle \{aba\}, \rho \rangle = \langle b, \{a\check{a}a\}^\wedge \rangle.$$

Now (6.11) follows from the definition (6.6). \square

Lemma 6.5. If $a \in A$ and $v \in A^{**}$ is a minimal idempotent, then

$$(6.12) \quad \{vav\} = \langle a, \hat{v} \rangle v.$$

Proof. Let $\pi: A \rightarrow B(H)_{sa}$ be an irreducible representation. Then $\tilde{\pi}(v)$ is a one-dimensional projection or zero. Hence

$$\tilde{\pi}(\{vav\}) = \tilde{\pi}(v)\pi(a)\tilde{\pi}(v) = \lambda\tilde{\pi}(v)$$

for some $\lambda \in \mathbb{R}$. We will show that $\lambda = \langle a, \hat{v} \rangle$. If $\tilde{\pi}(v) = 0$, there is nothing to prove. If $\tilde{\pi}(v) \neq 0$, then $\tilde{\pi}(v)$ is the projection onto $\text{lin}(\xi)$ where $\xi \in H$ is a vector representing \hat{v} (Lemma 6.3).

Now $\tilde{\pi}(v)\xi = \xi$, so we get

$$\begin{aligned}\langle a, \hat{v} \rangle &= (\pi(a)\xi | \xi) = (\tilde{\pi}(v)\pi(a)\tilde{\pi}(v)\xi | \xi) \\ &= (\tilde{\pi}(\{vav\})\xi | \xi) = \lambda(\tilde{\pi}(v)\xi | \xi) = \lambda\langle v, \hat{v} \rangle = \lambda.\end{aligned}$$

Since the irreducible representations separate points, this proves (6.12). \square

Lemma 6.6. If $a \in A$ and $\rho \in \partial_e K$ with $\langle a^2, \rho \rangle \neq 0$, then

$$(6.13) \quad (\check{\rho} \circ \check{\rho}_a)^\wedge = \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle T_a^* \rho,$$

where T_a denotes the Jordan multiplication operator $b \mapsto a \circ b$.

Proof. Let $\pi: A \rightarrow B(H)_{sa}$ be an irreducible representation associated with ρ . By Lemma 6.3, for $b \in A$:

$$(6.14) \quad \langle b, (\check{\rho} \circ \check{\rho}_a)^\wedge \rangle = \text{tr}(\tilde{\pi}(\check{\rho} \circ \check{\rho}_a)\pi(b)).$$

Note that the operator $\tilde{\pi}(\check{\rho}) \circ \pi(\check{\rho}_a)$ has at most two-dimensional range, so $\check{\rho} \circ \check{\rho}_a \in A_f$. Hence $(\check{\rho} \circ \check{\rho}_a)^\wedge$ is defined.

Using Lemma 6.4, the general Jordan identity $x \circ \{yxy\} = \{xyx\} \circ y$, and Lemma 6.5, we find

$$\begin{aligned}\check{\rho} \circ \check{\rho}_a &= \langle a^2, \rho \rangle^{-1} (\check{\rho} \circ \{a\check{\rho}a\}) \\ &= \langle a^2, \rho \rangle^{-1} (\{\check{\rho}a\check{\rho}\} \circ a) = \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle (\check{\rho} \circ a).\end{aligned}$$

Substitution into (6.14), use of the commutativity property of the trace, and application of Lemma 6.3 now gives

$$\begin{aligned}\langle b, (\check{\rho} \circ \check{\rho}_a)^\wedge \rangle &= \frac{1}{2} \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle \text{tr}(\tilde{\pi}(\check{\rho})\pi(a)\pi(b) + \pi(a)\tilde{\pi}(\check{\rho})\pi(b)) \\ &= \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle \text{tr}(\tilde{\pi}(\check{\rho})\pi(a \circ b)) = \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle \langle a \circ b, \rho \rangle.\end{aligned}$$

Since $\langle a \circ b, \rho \rangle = \langle b, T_a^* \rho \rangle$, this gives (6.13). \square

For the proof of the next proposition we note that in the middle term of formula (6.2) of Lemma 6.1 we must have $\frac{1}{2}\|(r-s)^2\| = \|[(r-s)^2]^\wedge\|$ since $\|\gamma(\rho, \sigma)\| = 1$. Since $[(r-s)^2]^\wedge \geq 0$, we also have $\|[(r-s)^2]^\wedge\| = \langle e, [(r-s)^2]^\wedge \rangle$. Hence (6.2) can be rewritten as follows

$$(6.15) \quad \gamma(\rho, \sigma) = \langle e, [(r-s)^2]^\wedge \rangle^{-1} [(r-s)^2]^\wedge.$$

Proposition 6.7. If $a \in A$, then $\rho \mapsto \gamma(\rho, \rho_a)$ is a w^* -continuous map from its domain $W_a \subseteq \partial_e K$ into K .

Proof. By (6.15) we must prove w^* -continuity of the map $\rho \mapsto [(\check{\rho} - \check{\rho}_a)^2]^\wedge$. Now

$$[(\check{\rho} - \check{\rho}_a)^2]^\wedge = (\check{\rho} - 2\check{\rho} \circ \check{\rho}_a + \check{\rho}_a)^2 = \rho - 2(\check{\rho} \circ \check{\rho}_a)^\wedge + \rho_a.$$

Hence it suffices to prove continuity of the map $\rho \mapsto (\check{\rho} \circ \check{\rho}_a)^\wedge$. But this is immediate from Lemma 6.6. \square

Definition. A collection \mathcal{S} of (proper) 3-balls $B(\rho, \sigma) \subseteq K$ with $\rho, \sigma \in \partial_e K$ and $\sigma \in F_\rho$, is said to be consistently oriented if the orientations are chosen such that for each $a \in A$ the map $\rho \mapsto \rho \times \rho_a$ is w^* -continuous whenever defined, i.e. for $\rho \neq \rho_a$ and $B(\rho, \rho_a) \in \mathcal{S}$. We say K is provided with a global orientation if the collection of all such balls in K is consistently oriented, and we say that K is orientable if it can be provided with a global orientation.

In the remaining part of this section we will relate the concept of orientability to the possibility of making a "good" choice of irreducible representations π_ρ associated with each pure state $\rho \in \partial_e K$.

Lemma 6.8. Let $a \in A$ and $\rho \in \partial_e K$ with $\langle a, \rho \rangle^2 < \langle a^2, \rho \rangle$, let $\pi : A \rightarrow B(H)_{sa}$ be an irreducible representation associated with ρ , and let $\xi \in H$ be a vector representing ρ . If the ball $B(\rho, \rho_a)$ is given the orientation induced by π , then for each $b \in A$:

$$(6.16) \quad \langle b, \rho \times \rho_a \rangle = \langle b, \gamma(\rho, \rho_a) \rangle + \frac{\langle a, \rho \rangle}{\langle a^2, \rho \rangle} (i[\pi(a), \pi(b)] \xi | \xi) .$$

Proof. Using Lemma 6.3, Lemma 6.4, the commutativity property of the trace, Lemma 6.5, and then Lemma 6.3 again, we find

$$\begin{aligned} & \langle b, \{(\pi|_A)^{-1}(i[\tilde{\pi}(\check{\rho}), \tilde{\pi}(\check{\rho}_a)])\}^\wedge \rangle \\ &= \text{tr}(i[\tilde{\pi}(\check{\rho}), \tilde{\pi}(\check{\rho}_a)]\pi(b)) \\ &= \langle a^2, \rho \rangle^{-1} \text{tr}(i[\tilde{\pi}(\check{\rho}), \pi(a)\tilde{\pi}(\check{\rho})\pi(a)]\pi(b)) \\ &= \langle a^2, \rho \rangle^{-1} \text{tr}(i\tilde{\pi}(\check{\rho})\pi(a)\pi(\check{\rho})[\pi(a), \pi(b)]) \\ &= \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle \text{tr}(i\tilde{\pi}(\check{\rho})[\pi(a), \pi(b)]) \\ &= \langle a^2, \rho \rangle^{-1} \langle a, \rho \rangle (i[\pi(a), \pi(b)] \xi | \xi) . \end{aligned}$$

By Proposition 6.2, this gives (6.16). \square

The next lemma contains somewhat more information than actually needed for the following proposition. However, it may be of some independent interest since it sheds light on the interplay between the "real but curved" geometry of the state space and the "flat but complex" geometry of the Hilbert spaces of the irreducible representations.

Lemma 6.9. Let $\pi : A \rightarrow B(H)_{sa}$ be an irreducible representation, and let $\xi, \eta \in H$ be unit vectors. Now, $\xi \perp \eta$ iff the states ρ, σ represented by ξ and η are antipodal in the (proper) 3-ball $B(\rho, \sigma)$. Moreover, if $\xi \perp \eta$ and $\zeta_t = (\cos t)\xi + (\sin t)\eta$, then as t runs from 0 to $\alpha \leq \frac{\pi}{2}$, the state τ_t represented by ζ_t will describe

a geodesic arc of 2α radians on the surface of $B(\rho, \sigma)$. In particular, $B(\rho, \sigma) = B(\rho, \tau_t)$ for all $0 < t < \frac{\pi}{2}$.

Proof. Note first that ρ and σ are antipodal in $B(\rho, \sigma)$ precisely when $\langle \check{\sigma}, \rho \rangle = 0$ (or equivalently $\langle \check{\rho}, \sigma \rangle = 0$, cf. [6 , § 4]). By Lemma 6.3, $\tilde{\pi}(\check{\sigma})$ is the projection onto $\text{lin}(\eta)$. Hence

$$\langle \check{\sigma}, \rho \rangle = (\tilde{\pi}(\check{\sigma})\xi | \xi) = ((\xi | \eta)\eta | \xi) = |(\xi | \eta)|^2 ,$$

which proves the first statement of the lemma. (Note that this also follows from general results of [4 ; § 11]).

Computing the right hand side of $\langle a, \tau_t \rangle = (\pi(a) \zeta_t | \zeta_t)$ for arbitrary $a \in A$, we find:

$$(6.17) \quad \tau_t = (\cos t)^2 \rho + (\sin t)^2 \sigma + (\sin 2t) \varphi ,$$

where $\varphi \in A^*$ is the (non-positive) functional $a \mapsto \text{Re}(\pi(a)\xi | \eta)$.

Consider now the vector $\zeta' = \zeta_{\frac{\pi}{4}}$ together with the state τ' which it represents. By (6.17) we have $\tau' = \omega + \varphi$, where $\omega = \frac{1}{2}(\rho + \sigma)$ is the center of the ball $B(\rho, \sigma)$. Now

$$\langle \check{\rho}, \tau' \rangle = \langle \check{\rho}, \omega \rangle + \text{Re}(\tilde{\pi}(\check{\rho})\xi | \eta) = \frac{1}{2} + \text{Re}(\xi | \eta) = \frac{1}{2} ,$$

which means that $\overrightarrow{\omega\rho}$ is orthogonal to $\overrightarrow{\omega\tau'}$ in the ball $B(\rho, \sigma)$.

Using the elementary formulae $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$ and $\sin^2 t = \frac{1}{2}(1 - \cos^2 t)$, and noting that $\frac{1}{2}(\rho - \sigma) = \rho - \omega$ and $\varphi = \tau' - \omega$, we can now transform (6.17) into:

$$\tau_t = \omega + (\cos 2t)(\rho - \omega) + (\sin 2t)(\tau' - \omega) ,$$

or otherwise stated:

$$\overrightarrow{\omega\tau_t} = (\cos 2t)\overrightarrow{\omega\rho} + (\sin 2t)\overrightarrow{\omega\tau'} .$$

This completes the proof. \square

For the proof of our next proposition we note that the usual proof of Kadison's transitivity theorem [26] can be applied in the present setting. Thus, if $\pi: A \rightarrow B(H)_{sa}$ is an irreducible representation and $\{\xi_1, \dots, \xi_n\}$ is a linearly independent set of vectors which can be mapped onto another set $\{\eta_1, \dots, \eta_n\}$ by an operator in $B(H)_{sa}$, then this can be done by an operator in $\pi(A)$, i.e. $\pi(a)\xi_j = \eta_j$ for all $j = 1, \dots, n$ and some $a \in A$ (In fact, Størmer has proved a general version of this theorem for an arbitrary JC-algebra [37]. In this generality the proof is more complicated since one also has to take into account the "real" and "quaternionic" cases which are eliminated by the 3-ball property in our setting).

Proposition 6.10. Let $\rho \in \partial_e K$, and assume that the balls $B(\sigma, \sigma') \subseteq F_\rho$, with $\sigma, \sigma' \in \partial_e F_\rho$, $\sigma \neq \sigma'$, are consistently oriented. Then the orientation of each of these balls is induced by one common irreducible representation π associated with ρ .

Proof: Let $\pi: A \rightarrow B(H)_{sa}$ be an irreducible representation which is associated with ρ and induces the given orientation of one arbitrarily chosen ball $B(\rho, \sigma)$ with $\sigma \in \partial_e F_\rho$, $\sigma \neq \rho$. Clearly, we shall be through if we can prove that π also induces the given orientation of any other ball $B(\rho, \sigma')$ with $\sigma' \in \partial_e F_\rho$, $\sigma' \neq \rho$.

Without loss of generality we can assume that ρ and σ are antipodal in $B(\rho, \sigma)$ and that ρ and σ' are antipodal in $B(\rho, \sigma')$. Let ρ, σ, σ' be represented by the unit vectors $\xi, \eta, \eta' \in H$, respectively. Now, $\xi \perp \eta$ and $\xi \perp \eta'$ by Lemma 6.9. We can assume (adjusting η' by a scalar factor of absolute value one if necessary) that $(\eta | \eta') = \cos \vartheta$ where $0 < \vartheta \leq \frac{\pi}{2}$. Then for a suitable unit vector $\xi \perp \eta$, we shall have $\eta' = (\cos \vartheta)\eta + (\sin \vartheta)\xi$. For each $t \in [0, \frac{\pi}{2}]$ we now write

$$(6.18) \quad \xi_t = (\cos t)\eta + (\sin t)\zeta,$$

and we let ρ_t be the state represented by the vector ξ_t . We will prove by a connectedness argument that the given orientation of each of the balls $B(\rho, \rho_t)$, with $t \in [0, \frac{\pi}{2}]$, is induced by π .

Observe first that (ξ, η, ζ) is an ortho-normal triple in H , and that there exists a bounded self-adjoint operator on H which maps ξ onto $\xi + \eta + \zeta$, η onto $\xi + \eta$, and ζ onto $\xi + \zeta$. Hence we can use the transitivity theorem to find $a \in A$ such that:

$$\pi(a)\xi = \xi + \eta + \zeta, \quad \pi(a)\eta = \xi + \eta, \quad \pi(a)\zeta = \xi + \zeta.$$

It follows that for each $t \in [0, \frac{\pi}{2}]$:

$$(6.19) \quad \pi(a)\xi_t = (\cos t + \sin t)\xi + \xi_t.$$

Note that by virtue of (6.7) the state $(\rho_t)_a$ is represented by the vector

$$(6.20) \quad \xi'_t = \|\pi(a)\xi_t\|^{-1} \pi(a)\xi_t.$$

By (6.19) the vector ξ'_t is located on the open quarter-circle joining ξ_t and ξ in H . Hence it follows from Lemma 6.9 that $B(\rho_t, \rho) = B(\rho_t, (\rho_t)_a)$ for each $t \in [0, \frac{\pi}{2}]$.

Now, let E be the set of all $t \in [0, \frac{\pi}{2}]$ for which the given orientation of $B(\rho_t, \rho)$ coincides with that induced by π , and let $\varepsilon(t) = +1$ for $t \in E$ and $\varepsilon(t) = -1$ for $t \notin E$. Also let \times denote the given orientation of each ball $B(\rho_t, \rho) = B(\rho_t, (\rho_t)_a)$ for $0 \leq t \leq \frac{\pi}{2}$. By Lemma 6.8, for each $b \in A$ and $t \in [0, \frac{\pi}{2}]$:

$$\langle b, \rho_t \times (\rho_t)_a \rangle = \langle b, \gamma(\rho_t, (\rho_t)_a) \rangle + \frac{\langle a, \rho_t \rangle}{\langle a^2, \rho_t \rangle} \varepsilon(t) (i[\pi(a), \pi(b)] \xi_t | \xi_t).$$

(Note that $0 < \langle a, \rho_t \rangle^2 < \langle a^2, \rho_t \rangle$ by virtue of (6.19)). By the con-

sistency hypothesis the left hand side of this equation is a continuous function of t , and by Proposition 6.7 the first term on the right hand side is also a continuous function of t . Hence the function:

$$(6.21) \quad t \mapsto \xi(t)(i[\pi(a), \pi(b)]\xi_t | \xi_t)$$

is continuous on $[0, \frac{\pi}{2}]$.

We now assume, for contradiction, that the set $F = [0, \frac{\pi}{2}] \setminus E$ is non-empty. Clearly $0 \in E$ so $E \neq \emptyset$. Hence, by the connectedness of $[0, \frac{\pi}{2}]$, there exists $t_0 \in \overline{E} \cap \overline{F}$. Observe now that there exists $b \in A$ such that

$$(6.22) \quad (i[\pi(a), \pi(b)]\xi_{t_0} | \xi_{t_0}) \neq 0.$$

(One possible choice of b is obtained by using the transitivity theorem to get $\pi(b)\xi_{t_0} = -i\xi$ and $\pi(b)\xi = i\xi_{t_0}$; then the left hand side of (6.22) will take the value $\cos t_0 + \sin t_0 > 0$).

Now it follows from the continuity of the function in (6.21) and the fact that $\xi(t)$ assumes both values ± 1 in any neighbourhood of t_0 , that the function

$$t \mapsto (i[\pi(a), \pi(b)]\xi_t | \xi_t)$$

must have a discontinuity for $t = t_0$. But this contradicts the explicit definition (6.18) of ξ_t . \square

Theorem 6.11. Let A be a JB-algebra whose state space K has the 3-ball property. Then K is orientable iff it is possible to choose irreducible representations $\pi_\rho : A \rightarrow B(H_\rho)$ (together with representing vectors $(\xi_\rho \in H_\rho)$ associated with the pure states $\rho \in \partial_e K$ such that

$$(6.23) \quad \rho \mapsto (i[\pi_\rho(a), \pi_\rho(b)]\xi_\rho | \xi_\rho)$$

is a w^* -continuous function on $\partial_e K$ for each pair $a, b \in A$. More specifically, the collection of all (proper) 3-balls $B(\rho, \sigma)$ is consistently oriented precisely when the orientations are induced by irreducible representations π_ρ satisfying this continuity requirement.

Proof. 1.) Assume first that K is orientable, and that the collection of all (proper) 3-balls $B(\rho, \sigma) \subseteq K$, with $\rho, \sigma \in \partial_e K$ and $\sigma \neq \rho, \sigma \in F_\rho$, has been consistently oriented. Now it follows from Proposition 6.10 that we can assign to each $\rho \in \partial_e K$ an irreducible representation π_ρ such that π_ρ induces the given orientation of each ball $B(\sigma, \sigma') \subseteq F_\rho$ with $\sigma, \sigma' \in \partial_e F_\rho$, $\sigma \neq \sigma'$. (Clearly, $F_\rho = F_\sigma$ implies unitary equivalence of π_ρ and π_σ).

Now, let $a, b \in A$ be arbitrary. Then we can apply formula (6.16) of Lemma 6.8 with π_ρ in place of π and ξ_ρ in place of ξ for each ρ in the set W_a (defined in (6.9)). By the consistency of the orientations the left hand side of (6.16) is a w^* -continuous function of ρ , and by Proposition 6.7 the first term on the right hand side of (6.16) is also a w^* -continuous function of ρ . Hence the function (6.23) must be w^* -continuous on its domain

$W'_a = \{\rho \in W_a \mid \langle a, \rho \rangle \neq 0\}$. Interchanging the roles of a and b , we also conclude that the function (6.23) must be w^* -continuous on W'_b .

In order to prove w^* -continuity of (6.23) for an arbitrary $\rho \in \partial_e K$, we first observe that we can assume without loss of generality that $\langle a, \rho \rangle, \langle b, \rho \rangle$ are non-zero, since we can add scalar multiples of the identity to a and b without changing the value of the right hand side of (6.23). Thus, by the remarks above, it only remains to prove w^* -continuity of (6.23) when $0 \neq \langle a, \rho \rangle^2 = \langle a^2, \rho \rangle$ and $0 \neq \langle b, \rho \rangle^2 = \langle b^2, \rho \rangle$. This means that ξ_ρ is an eigenvector

for both $\pi_\rho(a)$ and $\pi_\rho(b)$, and in this case $[\pi_\rho(a), \pi_\rho(b)]\xi_\rho = 0$. Thus we must show that $([\pi_\sigma(a), \pi_\sigma(b)]\xi_\sigma | \xi_\sigma)$ will be arbitrarily small for all σ in some w^* -neighbourhood of such a point $\rho \in \partial_e K$.

By the Cauchy-Schwartz inequality, and some elementary computation, we find for any $\sigma \in \partial_e K$:

$$\begin{aligned} |([\pi_\sigma(a), \pi_\sigma(b)]\xi_\sigma | \xi_\sigma)|^2 &\leq ([\pi_\sigma(a), \pi_\sigma(b)]^* [\pi_\sigma(a), \pi_\sigma(b)]\xi_\sigma | \xi_\sigma) \\ &= |\langle -2a \circ \{bab\} + \{ab^2a\} + \{ba^2b\}, \sigma \rangle|. \end{aligned}$$

The second term of this relation is seen to vanish for $\sigma = \rho$, and the last term is seen to be a w^* -continuous function of σ . This gives the desired continuity at the point ρ .

2.) Assume next that the representations π_ρ are chosen such that the function (6.23) is w^* -continuous for each pair $a, b \in A$. We must first show that if $\rho, \sigma \in \partial_e K$ and $\sigma \in F_\rho$ then π_ρ and π_σ are unitarily equivalent (and thus induce the same orientation on $B(\rho, \sigma)$). Since π_ρ is also associated with ρ , there is a unit vector ξ'_σ which represents the state σ with respect to the representation π_ρ . By Proposition 3.11

$$([\pi_\sigma(a), \pi_\sigma(b)]\xi_\sigma | \xi_\sigma) = \xi(\sigma)([\pi_\rho(a), \pi_\rho(b)]\xi'_\sigma | \xi'_\sigma),$$

where $\xi(\sigma) = +1$ if π_σ and π_ρ are unitarily equivalent and $\xi(\sigma) = -1$ if they are conjugate. Now, a connectedness argument as in the proof of Proposition 6.10 shows $\xi(\sigma) = 1$ for all $\sigma \in F_\rho$, so π_σ and π_ρ are unitarily equivalent.

Assume now that the balls $B(\rho, \sigma)$ with $\rho \neq \sigma$, $\sigma \in F_\rho$ are equipped with the orientations induced by π_ρ for $\rho, \sigma \in \partial_e K$. By Lemma 6.8 and Proposition 6,7, the map $\rho \mapsto \langle b, \rho \times \rho_a \rangle$ is w^* -continuous whenever defined, for each pair $a, b \in A$. Hence the balls are consistently oriented, so K is orientable. \square

Corollary 6.12. The state space K of any C^* -algebra \mathcal{A} is orientable. Specifically, the usual GNS-representations of \mathcal{A} will induce a global orientation of K .

Proof. For each $\rho \in \partial_e K$ let π_ρ be the usual GNS-representation associated with ρ . Then for each pair $a, b \in \mathcal{A}_{sa}$:

$$(i[\pi_\rho(a), \pi_\rho(b)]\xi_\rho | \xi_\rho) = (i\pi_\rho([a, b])\xi_\rho | \xi_\rho) = \langle i[a, b], \rho \rangle.$$

Now the corollary follows from Theorem 6.11. \square

In the sequel we will use the term oriented state space for a C^* -algebra to denote the state space together with the global orientation mentioned in Corollary 6.12.

Remark. Our definition of a global orientation involves the maps $\rho \mapsto \rho_a$ of $\partial_e K$. Since these maps are given in terms of the Jordan triple product, our definition depends on the Jordan structure of A . However, the Jordan product in A is uniquely determined by the structure of A as an order-unit space, cf. [4; Th.12.13]. Hence it is clear a priori that the maps $\rho \mapsto \rho_a$ can be described in terms of notions pertaining to the compact convex set K without any reference to the Jordan product in A . We will now explain one way in which this can be done. (Since we shall not need this characterization later on, we will omit the details of the proof).

Note first that for the definition of a global orientation we only need to characterize the maps $\rho \mapsto \rho_a$ in the case where a is positive and invertible. In fact the proof of Theorem 6.11 establishes the equivalence of w^* -continuity for the maps

$$(6.24) \quad \rho \mapsto (i[\pi_\rho(a), \pi_\rho(b)]\xi_\rho | \xi_\rho)$$

and

$$(6.25) \quad \rho \rightarrow \langle b, \rho \times \rho_a \rangle$$

and the map (6.24) is clearly unaffected if a multiple of e is added to a .

By an automorphism of the cone $(A^*)^+$ generated by K we understand a w^* -bicontinuous affine map of $(A^*)^+$ onto itself. Note that an automorphism of $(A^*)^+$ is uniquely extendable to a w^* -bicontinuous linear order automorphism of A^* . Recall also that there exists an inner product in $\text{lin } \partial_e K$ such that $(\rho|\sigma) = \langle \check{\rho}, \sigma \rangle$ for $\rho, \sigma \in \partial_e K$ [6; Lemmas 5.5, 6.4]. Consider now a linear map $S: A^* \rightarrow A^*$ leaving $\text{lin } \partial_e K$ invariant, and note that S is symmetric with respect to this inner product iff

$$(6.26) \quad \langle \check{\rho}, S\sigma \rangle = \langle (S\rho)^\vee, \sigma \rangle$$

for all $\rho, \sigma \in \partial_e K$, and that S is positive semi-definite if in addition

$$(6.27) \quad \sum_{i=1}^n \langle \check{\rho}_i, S\rho_j \rangle \lambda_i \lambda_j \geq 0$$

for all $\rho_1, \dots, \rho_n \in \partial_e K$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Note in particular that if S is an automorphism of $(A^*)^+$ then S preserves extreme rays, so it leaves $\text{lin } \partial_e K$ invariant. Hence, the above requirements make sense for such an S .

Now the desired characterization states that a map S of $(A^*)^+$ onto itself is the dual of a map of the form $b \mapsto \{aba\}$ for some positive and invertible a iff S is an automorphism of $(A^*)^+$ which is positive semi-definite in the sense of (6.26) and (6.27). (In fact, when S is an automorphism of $(A^*)^+$, then (6.27) implies the similar relation with strict inequality. Hence, we could as

well have written "definite" in place of "semi-definite in this characterization).

The proof of this result is based on arguments similar to those in the proof of [6; Lem.6.4]. At one point in the proof one has to show that an operator is bounded in the inner product norm of $\text{lin } \partial_e K$. A simple proof of this can be obtained by appealing to a lemma of Woronowicz [42; Lem.2]).

§ 7. The main theorem

We are now ready to characterize those state spaces of JB-algebras which are affinely homeomorphic to state spaces of C^* -algebras. This, when combined with the results of [6], gives a characterization of those compact convex sets which are affinely homeomorphic to state spaces of C^* -algebras.

We remark that one can have a JC-algebra $A \subseteq B(H)_{sa}$ which is not the self-adjoint part of any C^* -subalgebra of $B(H)$, although A admits some faithful representation $\pi: A \rightarrow B(H')_{sa}$ such that $\pi(A)$ is the self-adjoint part of a C^* -subalgebra of $B(H')$. (Example: take the self-adjoint part A of any non-abelian C^* -algebra, and realize it as a JC-algebra by the canonical embedding into the enveloping C^* -algebra.) The following theorem which is our main result, describes how one can choose a representation $\pi: A \rightarrow B(H)_{sa}$ for a JB-algebra with an appropriate state space so that $\pi(A) + i\pi(A)$ is a C^* -algebra.

Theorem 7.1. A JB-algebra A is isomorphic to the self-adjoint part of a C^* -algebra iff its state space K has the 3-ball property and is orientable. Specifically, if we choose the irreducible representations $\pi_\rho: A \rightarrow B(H_\rho)_{sa}$ associated with the pure states $\rho \in \partial_e K$ in correspondence with a global orientation of K , or equivalently such that the function

$$(7.1) \quad \rho \mapsto (i[\pi_\rho(a), \pi_\rho(b)]\xi_\rho | \xi_\rho)$$

(where $\xi_\rho \in H$ represents ρ) is w^* -continuous on $\partial_e K$ for each pair $a, b \in A$, then $\pi = \bigoplus_\rho \pi_\rho$ will map A onto the self-adjoint part of a C^* -algebra, namely $\pi(A) + i\pi(A)$. This will not be the case if the representations π_ρ are not chosen in correspondence with a global orientation.

Proof. If A is isomorphic to the self-adjoint part of a C^* -algebra, then K has the 3-ball property by Proposition 3.3 and is orientable by Corollary 6.12.

Conversely we assume that K has these properties and that π_ρ have been chosen for all $\rho \in \partial_e K$ corresponding to some global orientation of K . By Theorem 6.11 this is equivalent to the continuity of the function (7.1) for given $a, b \in A$, as stated in the theorem. Let \mathcal{O} be the enveloping C^* -algebra of A with state space \mathcal{K} . We define X_1 to be the set of all $\rho \in \partial_e \mathcal{K}$ for which $\pi_\rho|_A$ is unitarily equivalent to the GNS-representation Π_ρ of \mathcal{O} restricted to A , and we define X_2 in the same manner with "conjugate" replacing "unitarily equivalent".

Note that by Propositions 5.6 and 3.9, $X_1 \cup X_2 = \partial_e \mathcal{K}$ and $X_1 \cap X_2 = \partial_{e,0} \mathcal{K}$. By Proposition 5.6 we also have $\Phi^*(X_1) = X_2$ and $\Phi^*(X_2) = X_1$. Furthermore, by Proposition 3.11 for $\rho \in \partial_e \mathcal{K}$ then $\rho \in X_1$ iff:

$$(7.2) \quad ([\Pi_\rho(a), \Pi_\rho(b)] \xi_\rho | \xi_\rho) = ([\pi_\rho(a), \pi_\rho(b)] \xi_\rho | \xi_\rho) \quad \text{all } a, b \in A.$$

Note that the map

$$(7.3) \quad \rho \mapsto \langle [a, b], \rho \rangle = ([\Pi_\rho(a), \Pi_\rho(b)] \xi_\rho | \xi_\rho)$$

is w^* -continuous on $\partial_e \mathcal{K}$ for all $a, b \in \mathcal{O}$, and note also that the map

$$(7.4) \quad \rho \mapsto ([\pi_{\rho|_A}(a), \pi_{\rho|_A}(b)] \xi_\rho | \xi_\rho)$$

is w^* -continuous on $\partial_e \mathcal{K}$ by the continuity of the map (7.1) and by the continuity of the restriction map $\rho \mapsto \rho|_A$. Now it follows from (7.2) and from the continuity of (7.3) and (7.4) that X_1 is w^* -closed in $\partial_e \mathcal{K}$. Also X_2 is w^* -closed in $\partial_e \mathcal{K}$ since X_1 and X_2 are interchanged by Φ^* .

Observe that since $X_1 \cap X_2 = \partial_{e,o} \mathcal{K}$, then $\partial_{e,1} \mathcal{K}$ is the disjoint union of $X_1 \cap \partial_{e,1} \mathcal{K}$ and $X_2 \cap \partial_{e,1} \mathcal{K}$. These are both closed in $\partial_{e,1} \mathcal{K}$ and so they are also open in $\partial_{e,1} \mathcal{K}$. By the 3-ball property, $\partial_e F_\rho$ is path connected for each $\rho \in \partial_e \mathcal{K}$. It follows that $\partial_e F_\rho \subseteq X_1$ whenever $\rho \in X_1$.

Now define

$$F_1 = \overline{\text{co}}(X_1) = \overline{\text{co}}\left(\bigcup_{\rho \in X_1} F_\rho\right)$$

Now F_1 is a split face of \mathcal{K} by virtue of [2 ; Lem.7.2] (or by [38; Cor.5.3] where a related result is stated without explicitly mentioning split faces). In the same manner we introduce the w^* -closed split face $F_2 = \overline{\text{co}}(X_2)$. Note further that by Milman's theorem $\partial_e F_1 \subseteq \overline{X_1} = X_1$, so $\partial_e F_1 = X_1$. Similarly $\partial_e F_2 = X_2$. Note also that Φ^* exchanges F_1 and F_2 , and that $\partial_e(F_1 \cap F_2) = X_1 \cap X_2 = \partial_{e,o} \mathcal{K}$.

Next we show that the restriction map sends F_1 bijectively onto K . For surjectivity it suffices to show that $\partial_e F_1$ restricts onto $\partial_e K$ (by w^* -continuity and the Krein-Milman theorem). Let $\sigma \in \partial_e K$ be given; by Proposition 5.6 there exists $\rho \in \partial_e \mathcal{K}$ such that $\rho|_A = \sigma$. Now since $X_1 \cup X_2 = \partial_e \mathcal{K}$, then either $\rho \in X_1 = \partial_e F_1$, or else $\rho \in X_2$ so $\Phi^* \rho \in X_1 = \partial_e F_1$ and $(\Phi^* \rho)|_A = \sigma$.

To show the restriction map is injective, we begin by defining

$$G_1 = F_1 \cap F'_2, \quad G_{12} = F_1 \cap F_2, \quad G_2 = F_2 \cap F'_1.$$

Note that G_1, G_{12}, G_2 are disjoint split faces of \mathcal{K} with $\text{co}(G_1 \cup G_{12}) = F_1$ and $\text{co}(G_2 \cup G_{12}) = F_2$. Now Φ^* will interchange G_1 and G_2 , and it will fix everything in G_{12} since $\partial_e G_{12} = \partial_{e,o} \mathcal{K}$. Let $\rho, \sigma \in F_1$ and assume $\rho|_A = \sigma|_A$. Observe

that the range of $I + \Phi$ is fixed by Φ , so $(I+\Phi)\mathcal{O}_{sa} = A$ by Lemma 5.4. Therefore

$$(7.5) \quad (I+\Phi)^*\rho = (I+\Phi)^*\sigma.$$

Now write

$$(7.6) \quad \rho = \alpha\rho_1 + (1-\alpha)\rho_{12}, \quad \sigma = \beta\sigma_1 + (1-\beta)\sigma_{12},$$

where $0 \leq \alpha, \beta \leq 1$, $\rho_1, \sigma_1 \in G_1$ and $\rho_{12}, \sigma_{12} \in G_{12}$. Thus from (7.5) and (7.6) we get

$$(7.7) \quad \begin{aligned} \alpha\rho_1 + (1-\alpha)\rho_{12} + \alpha\Phi^*\rho_1 + (1-\alpha)\rho_{12} \\ = \beta\sigma_1 + (1-\beta)\sigma_{12} + \beta\Phi^*\sigma_1 + (1-\beta)\sigma_{12}. \end{aligned}$$

Every member of $\mathcal{K} = \text{co}(G_1 \cup G_{12} \cup G_2)$ can be expressed uniquely as a convex combination of elements from G_1, G_{12}, G_2 (cf. e.g. [1 Prop II, 6.6]), so (7.7) implies $\alpha\rho_1 = \beta\sigma_1$ and $(1-\alpha)\rho_{12} = (1-\beta)\sigma_{12}$; therefore $\rho = \sigma$. Thus we have shown that the restriction map is a bijection of F_1 onto K .

Now define:

$$(7.8) \quad \Pi = \bigoplus_{\rho \in \partial_e F_1} \sum \Pi_\rho : \mathcal{O} \mapsto \bigoplus_{\rho \in \partial_e F_1} \sum B(H_\rho).$$

Note that

$$(\ker \Pi)^\perp = \left(\bigcap_{\rho \in \partial_e F_1} \ker \Pi_\rho \right)^\perp = F_1,$$

since F_1 is the smallest w^* -closed split face containing all $\overline{F}_\rho = (\ker \Pi_\rho)^\perp$. The annihilator $(F_1)_0 = \ker \Pi$ meets A in $\{0\}$, so Π will be faithful on A .

We claim $\Pi(A) = \Pi(\mathcal{O})_{sa}$. Suppose not; then since $\Pi(A)$ is norm closed by [6 ; Lem. 9.3], there will exist $\rho \in \Pi(\mathcal{O})_{sa}^*$ such that ρ is zero on $\Pi(A)$ but not on $\Pi(\mathcal{O})_{sa}$. Write $\rho = \rho_1 - \rho_2$ where ρ_1 and ρ_2 are positive and in $\Pi(\mathcal{O})_{sa}^*$; then $\rho_1 = \rho_2$

on $\Pi(A)$. Without loss we may assume $\rho_1(1) = \rho_2(1) = 1$. Now $\rho_1 \circ \Pi$ and $\rho_2 \circ \Pi$ are states in $(\ker \Pi)^\perp = F_1$ which agree on A . By the result above, $\rho_1 \circ \Pi = \rho_2 \circ \Pi$, and so $\rho = 0$. This proves $\Pi(\mathcal{OL})_{sa} = \Pi(A)$, and so $\Pi(A)$ is the self-adjoint part of a C^* -algebra. By choice of ρ , $\Pi_\rho|_A$ is unitarily equivalent to $\pi_\rho|_A$ for each $\rho \in \partial_e \mathcal{K}$, hence $\Pi|_A$ is unitarily equivalent to $\pi = \bigoplus_{\rho \in \partial_e K} \pi_\rho$. Hence $\pi(A)$ is the self-adjoint part of a C^* -algebra as claimed in the theorem.

Finally, suppose that for each $\rho \in \partial_e K$ we have chosen an irreducible representation π_ρ associated with ρ in such a way that $\bigoplus_\rho \pi_\rho(A)$ is the self-adjoint part of a C^* -algebra. Then for each pair $a, b \in A$ there exists $c \in A$ such that

$$\bigoplus_\rho i[\pi_\rho(a), \pi_\rho(b)] = \bigoplus_\rho \pi_\rho(c),$$

and so

$$i[\pi_\rho(a), \pi_\rho(b)] = \pi_\rho(c)$$

for each $\rho \in \partial_e K$. Now for each pair $a, b \in A$, the map

$$\rho \mapsto (i[\pi_\rho(a), \pi_\rho(b)] \xi_\rho | \xi_\rho) = \langle c, \rho \rangle$$

is w^* -continuous on $\partial_e K$. Hence the representations π_ρ are chosen in correspondence with a global orientation of K , by Theorem 6.11. The proof is complete. \square

Corollary 7.2. A compact convex set K (in a locally convex Hausdorff space) is affinely homeomorphic to the state space of a C^* -algebra iff:

- (i) every norm exposed face of K is projective,
- (ii) every $a \in A(K)$ admits an orthogonal decomposition
 $a = a^+ - a^-$ with $a^+, a^- \in A(K)^+$ and $a^+ \perp a^-$,
- (iii) the σ -convex hull of $\partial_e K$ is a split face,

- (iv) $B(\rho, \sigma)$ is a norm exposed face affinely isomorphic to a 3-ball or a line segment for each pair of distinct points $\rho, \sigma \in \partial_e K$.
- (v) K is orientable.

Proof. By [6; Th.7.2] (i), (ii), (iii), (iv) imply that K is affinely homeomorphic to the state space of a JB-algebra, or what is the same, that $A = A(K)$ can be organized to a JB-algebra with an appropriate Jordan product (which will be uniquely determined); conversely, if A is a JB-algebra, then (i), (ii), (iii) hold, and (iv) holds in the weaker form that each $B(\rho, \sigma)$ is a Hilbert ball of arbitrary dimension (finite or infinite). Now the corollary follows from Theorem 7.1. \square

Unlike JB-algebras, the C^* -algebras are not completely determined by their state spaces. However, the situation changes if we consider the state spaces together with global orientations, as we will now show.

Definition. Let A be a JB-algebra admitting a faithful representation as the self-adjoint part of a C^* -algebra \mathcal{O} (i.e. the state space of A has the 3-ball property and is orientable); then we say two such representations $\pi_i : A \rightarrow (\mathcal{O}_i)_{sa}$, $i = 1, 2$, are equivalent if there exists a $*$ -isomorphism Φ of \mathcal{O}_1 onto \mathcal{O}_2 such that $\pi_2 = \Phi \circ \pi_1$; an equivalence class of such representations will be called a C^* -structure on A .

If A is as above, then each faithful representation $\pi : A \rightarrow \mathcal{O}_{sa}$, where \mathcal{O} is a C^* -algebra, determines a "complex Lie product":

$$(7.9) \quad a, b \mapsto \pi^{-1}(i[\pi(a), \pi(b)]) .$$

Clearly, two such representations are equivalent iff they determine the same product (7.9) in A . Thus, to specify a C^* -structure is the same as to choose one of the possible complex Lie products (7.9).

Corollary 7.3. If A is a JB-algebra whose state space K has the 3-ball property and is orientable, then the C^* -structures on A are in 1-1 correspondence with the global orientations of K .

Proof. If $\pi: A \rightarrow \mathcal{O}_{sa}$ is a faithful representation of A onto the self-adjoint part of a C^* -algebra \mathcal{O} , then π^* is an affine homeomorphism of the state space of \mathcal{O} onto K . If σ is any pure state on \mathcal{O} and π_σ is the GNS-representation associated with σ , then $\pi_\sigma \circ \pi$ is an irreducible representation of A associated with the state $\rho = \pi^*(\sigma)$. Now let ξ_ρ be a vector representing the state σ with respect to π_σ and then also the state ρ with respect to $\pi_\sigma \circ \pi$. Then the function

$$(7.10) \quad \rho \mapsto \langle i[\pi(a), \pi(b)], \sigma \rangle = (i[\pi_\sigma(\pi(a)), \pi_\sigma(\pi(b))] \xi_\rho | \xi_\rho)$$

is w^* -continuous on $\partial_e K$ for each pair $a, b \in A$; hence the irreducible representations $\pi_\sigma \circ \pi$ determine a global orientation of K (cf. Theorem 6.11).

Clearly, by Proposition 3.11, two equivalent representations $\pi_i: A \rightarrow (\mathcal{O}_i)_{sa}$, $i = 1, 2$, will determine the same global orientations in this way; whereas two inequivalent representations will define complex Lie products in A which are different (i.e. of opposite sign), thus for some $\rho \in K$ the right hand side of (7.10) must assume opposite non-zero values for the two representations, and so they determine different global orientations. \square

Note that by Corollary 7.3 the oriented state space of a C^* -algebra is a dual object which has sufficiently rich structure to permit one to recapture the given C^* -algebra.

In order to apply Corollary 7.3 to recapture C^* -algebras from their dual objects, one needs practical methods to specify global orientations. For this purpose it is often convenient to proceed as in the proof of Theorem 7.1. Below, we have extracted the relevant information as a separate corollary for later references.

Corollary 7.4. Let A be a JB-algebra whose state space K has the 3-ball property, and let \mathcal{O} be the enveloping C^* -algebra with state space \mathcal{K} . Then there is a 1-1 correspondence between the global orientations of K (if any) and the w^* -closed split faces F of \mathcal{K} satisfying

$$(7.11) \quad \text{co}(F \cup \Phi^*(F)) = \mathcal{K} \quad , \quad F \cap \Phi^*(F) = \overline{\text{co}}(\partial_{e,o}\mathcal{K}) \quad .$$

Specifically, for a given global orientation of K , F is the w^* -closed hull of those $\rho \in \partial_e \mathcal{K}$ for which the associated GNS-representation Π_ρ of \mathcal{O} is one-dimensional or else induces the given orientation of the balls in $F_{r(\rho)}$.

§ 8. The canonical \mathbb{Z}_2 -bundles over the pure states, the spectrum and the structure space.

Following [21], we use the following:

Definition. A \mathbb{Z}_2 -bundle is a continuous surjection $p: X \rightarrow B$ of a topological space X onto a topological space B , with a \mathbb{Z}_2 -action on X by fiber-preserving homomorphisms in such a way that X/\mathbb{Z}_2 is homeomorphic to B . We say that the \mathbb{Z}_2 -bundle is trivial if it is isomorphic (in the natural sense) to the product \mathbb{Z}_2 -bundle $B \times \mathbb{Z}_2 \rightarrow B$.

Note that we do not require any condition of local triviality, and in general the \mathbb{Z}_2 -bundles we consider will not be locally trivial.

Throughout this section we consider a JB-algebra A whose state space K has the 3-ball property, and we denote the enveloping C^* -algebra and its state space by \mathcal{O} and \mathcal{K} . We are going to show that for the action of $\mathbb{Z}_2 = \{I, \hat{\sigma}^*\}$, then $\partial_{e,1}\mathcal{K} \rightarrow \partial_{e,1}K$, $\hat{\mathcal{K}}_1 \rightarrow \hat{K}_1$, and $\text{Prim}_1\mathcal{K} \rightarrow \text{Prim}_1K$ are \mathbb{Z}_2 -bundles. We will see that orientability of K is equivalent to triviality of each of these bundles.

Lemma 8.1. The restriction map $r: \mathcal{K} \rightarrow K$ maps (w*-closed) split faces of \mathcal{K} onto (w*-closed) split faces of K ; also the inverse image by r of each (w*-closed) split face of K will be a (w*-closed) split face of \mathcal{K} .

Proof. 1.) Note that r is the dual map of the canonical injection of A in \mathcal{O}_{sa} . Hence the first statement of the lemma follows from Proposition 2.10.

2.) Let F be a split face of K and let $c \in A^{**}$ be the corresponding central idempotent, i.e.

$$(8.1) \quad F = \{\rho \in K \mid \langle c, \rho \rangle = 1\}.$$

Assuming that \mathcal{O} is given in its universal representation, we can identify \overline{A} and A^{**} (cf. [18; proof of Th.1]). Since \mathcal{O} is generated by A (as a C^* -algebra) then $A'' = \mathcal{O}^{**}$. Thus, c will also be a central idempotent of \mathcal{O}^{**} . Now the inverse image $r^{-1}(F)$ consists of all $\rho \in \mathcal{K}$ such that $\langle c, \rho \rangle = 1$; hence F is a split face of \mathcal{K} . Clearly $r^{-1}(F)$ is w^* -closed if F is, so the proof is complete. \square

Corollary 8.2. For each $\rho \in \mathcal{K}$ we have $r(F_\rho) = F_{r(\rho)}$ and $r(\overline{F}_\rho) = \overline{F}_{r(\rho)}$.

Proof. By Lemma 8.1, $r(F_\rho) \supseteq F_{r(\rho)}$ and $r^{-1}(F_{r(\rho)}) \supseteq F_\rho$, so $r(F_\rho) = F_{r(\rho)}$. The second statement follows in a similar fashion.

Proposition 8.3. The restriction map $r: \mathcal{K} \rightarrow K$ determines continuous maps (which we also denote by r by abuse of notation) from $\partial_{e,1}\mathcal{K}$ onto $\partial_{e,1}K$ (where both spaces are equipped with the facial topology), from $\hat{\mathcal{K}}_1$ onto \hat{K}_1 , and from $\text{Prim}_1\mathcal{K}$ onto Prim_1K . In each of these three cases, the topology of the second space is the quotient topology transferred from the first.

Proof. By Corollary 8.2 and Proposition 5.6, r maps $\partial_{e,1}\mathcal{K}$ onto $\partial_{e,1}K$, $\hat{\mathcal{K}}_1$ onto \hat{K}_1 , and $\text{Prim}_1\mathcal{K}$ onto Prim_1K . To see that r is continuous and that the second spaces have the quotient topology, it suffices to show that r is a continuous and closed map in each case. By Lemma 8.1, r will determine a 1-1 correspondence between the Φ^* -invariant w^* -closed split faces of \mathcal{K} and the w^* -closed split faces of K . (The inverse image of a w^* -closed split face of K is Φ^* -invariant, as can be seen by considering

extreme points). By the definition of the topologies, this completes the proof. \square

Lemma 8.4. If $F, G \in \hat{\mathcal{K}}$ (or if $F, G \in \text{Prim } \mathcal{K}$) and if $r(F) = r(G)$, then either $F = G$ or $F = \Phi^*(G)$.

Proof. Let F and G be as announced, and consider $\rho \in \partial_e F$. We will first show there exists $\sigma \in \partial_e G$ with $r(\sigma) = r(\rho)$. Let $H = r^{-1}(r(\rho)) \cap G$. Then H is a non-empty face of G . If $F, G \in \hat{\mathcal{K}}$ then G is the σ -convex hull of its extreme points [16; Th. I.4.1], and it follows that the norm closed face H must satisfy $H \cap \partial_e G \neq \emptyset$. If instead $F, G \in \text{Prim } \mathcal{K}$, then H is a w^* -closed face of G , so by Krein-Milman $H \cap \partial_e G \neq \emptyset$. In either case we have a point $\sigma \in H \cap \partial_e G$, which will satisfy the requirement $r(\sigma) = r(\rho)$.

Henceforth, let $\sigma \in \partial_e G$ be fixed with $r(\sigma) = r(\rho)$. By Proposition 5.9 either $\sigma = \rho$ or $\sigma = \Phi^*(\rho)$.

Observe now that it suffices to show that the hypotheses of the lemma imply $F \subseteq G$ or $\Phi^*(F) \subseteq G$, since by symmetry this also gives $G \subseteq F$ or $\Phi^*(G) \subseteq F$ and since an examination of the four possible combinations shows that either $F = G$ or $F = \Phi^*(G)$.

Assume that $F = F_\rho$ in case $F, G \in \hat{\mathcal{K}}$, and that $F = \bar{F}_\rho$ in case $F, G \in \text{Prim } \mathcal{K}$. Since $\sigma \in \partial_e G$, we have $F_\sigma = G$ in the first case and $\bar{F}_\sigma \subseteq G$ in the second. In either case, we get $F \subseteq G$ when $\sigma = \rho$ and $\Phi^*(F) \subseteq G$ when $\sigma = \Phi^*(\rho)$. This completes the proof. \square

Proposition 8.5. For the action of $\mathbb{Z}_2 = \{I, \Phi^*\}$, $\partial_{e,1} \mathcal{K} \rightarrow \partial_{e,1} K$ (with the facial topologies), $\hat{\mathcal{K}}_1 \rightarrow \hat{K}_1$, and $\text{Prim}_1 \mathcal{K} \rightarrow \text{Prim}_1 K$ are all \mathbb{Z}_2 -bundles.

Proof. Set-theoretically the second spaces are the quotients of the first under the action of $\{I, \Phi^*\}$ by virtue of Proposition 5.9 and Lemma 8.4. The second spaces have the quotient topologies by Proposition 8.3. \square

Remark. By Corollary 5.10 each fiber of $\partial_{e,1}\mathcal{K} \rightarrow \partial_{e,1}K$, and of $\hat{\mathcal{K}}_1 \rightarrow \hat{K}_1$, will have two elements; we do not know if this holds for $\text{Prim}_1\mathcal{K} \rightarrow \text{Prim}_1K$.

Theorem 8.6. Let A be a JB-algebra whose state space K has the 3-ball property and let \mathcal{O} be the enveloping C^* -algebra with state space \mathcal{K} . Then the following are equivalent:

- (i) $\partial_{e,1}\mathcal{K} \rightarrow \partial_{e,1}K$ is a trivial \mathbb{Z}_2 -bundle,
- (ii) $\hat{\mathcal{K}}_1 \rightarrow \hat{K}_1$ is a trivial \mathbb{Z}_2 -bundle,
- (iii) $\text{Prim}_1\mathcal{K} \rightarrow \text{Prim}K$ is a trivial \mathbb{Z}_2 -bundle,
- (iv) K is orientable.

If these equivalent conditions hold, then the (continuous) cross-sections of each of the three \mathbb{Z}_2 -bundles are in 1-1 correspondence with the global orientations of K .

Proof. We will prove (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iv). Note first that by Corollary 7.4 statement (iv) is equivalent to:

- (iv)' There is a w^* -closed split face F of \mathcal{K} such that $\text{co}(F \cup \Phi^*(F)) = \mathcal{K}$ and $F \cap \Phi^*(F) = \overline{\text{co}} \partial_{e,0}\mathcal{K}$.

Assume first that (iv)' holds, and define

$$\begin{aligned} \mathcal{N}_1 &= \{\bar{F}_\rho \in \text{Prim}_1 \mid \bar{F}_\rho \subseteq F\}, \\ \mathcal{N}_2 &= \{\bar{F}_\rho \in \text{Prim}_1 \mid \bar{F}_\rho \subseteq \Phi^*(F)\}. \end{aligned}$$

Clearly, $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, $\mathcal{N}_1 \cup \mathcal{N}_2 = \text{Prim}_1 \mathcal{K}$, and by definition of the topology both \mathcal{N}_1 and \mathcal{N}_2 are closed in Prim_1 ; then they are also open in $\text{Prim}_1 \mathcal{K}$. Moreover, $\Phi^*(\mathcal{N}_i) = \mathcal{N}_j$ for $i \neq j$. By Lemma 8.4, the restriction map sends each one of $\mathcal{N}_1, \mathcal{N}_2$ bijectively onto $\text{Prim}_1 K$. By Proposition 8.3, the restriction map is then a homeomorphism of each one of $\mathcal{N}_1, \mathcal{N}_2$ onto $\text{Prim}_1 K$. This shows that $\text{Prim}_1 \mathcal{K} \rightarrow \text{Prim}_1 K$ is trivial.

Assume next that (iii) holds, i.e. that $\text{Prim}_1 \mathcal{K} \rightarrow \text{Prim}_1 K$ is trivial. Then we can choose two closed and open sets \mathcal{N}_1 and \mathcal{N}_2 in $\text{Prim}_1 \mathcal{K}$ with the same properties as above, i.e. $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, $\mathcal{N}_1 \cup \mathcal{N}_2 = \text{Prim}_1 \mathcal{K}$, the restriction map sends each of them bijectively onto $\text{Prim}_1 K$, and $\Phi^*(\mathcal{N}_i) = \mathcal{N}_j$ for $i \neq j$. Now define

$$\mathcal{M}_i = \{F_\rho \in \hat{\mathcal{K}}_1 \mid \overline{F}_\rho \in \mathcal{N}_i\}, \quad i = 1, 2.$$

Then \mathcal{M}_i is the inverse image in $\hat{\mathcal{K}}_1$ of \mathcal{N}_i for the map $F_\rho \rightarrow \overline{F}_\rho$ ($i=1,2$). It follows that each one of $\mathcal{M}_1, \mathcal{M}_2$ is closed and open in $\hat{\mathcal{K}}_1$, $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$, $\mathcal{M}_1 \cup \mathcal{M}_2 = \hat{\mathcal{K}}_1$, the restriction map sends each of them bijectively onto $\hat{\mathcal{K}}_1$ and $\Phi(\mathcal{M}_i) = \mathcal{M}_j$ for $i \neq j$. Thus, $\text{Prim}_1 \hat{\mathcal{K}} \rightarrow \text{Prim}_1 K$ is trivial.

Assuming that (ii) holds, we can find $\mathcal{M}_1, \mathcal{M}_2 \in \hat{\mathcal{K}}$ with the same properties as above. Pulling back \mathcal{M}_1 and \mathcal{M}_2 to $\partial_{e,1} \mathcal{K}$ by the map $\rho \mapsto F_\rho$ and arguing as above, we can now show that $\partial_{e,1} \mathcal{K} \rightarrow \partial_{e,1} K$ is trivial.

To close the circle of implications, we assume that (i) holds; then we can choose two sets $E_1, E_2 \subseteq \partial_e \mathcal{K}_1$ which are closed and open in the facial topology such that $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = \partial_e \mathcal{K}_1$, the restriction map sends each of them bijectively onto $\partial_e K_1$, and $\Phi(E_i) = E_j$ for $i \neq j$. Define now

$$F = \overline{\text{co}}(E_1 \cup \partial_{e,0} \mathcal{K}).$$

Note that by Proposition 2.11 $\partial_{e,o} \mathcal{K}$ is facially closed, so $\partial_{e,1} \mathcal{K}$ is facially open. Thus $E_1 \cup \partial_{e,o} \mathcal{K}$, which is equal to $\partial_e K \setminus E_2$, must be facially closed. Hence F is a w^* -closed split face such that $\partial_e F = E_1 \cup \partial_{e,o} \mathcal{K}$. Clearly, F has the properties required in (iv)', so K is orientable.

To prove the final statement of the theorem, it suffices to establish a 1-1 correspondence between (continuous) cross sections of the bundles and decompositions of the type $(\mathcal{N}_1, \mathcal{N}_2)$, $(\mathcal{M}_1, \mathcal{M}_2)$ (E_1, E_2) , discussed above. We will do this for $\text{Prim}_1 \mathcal{K} \rightarrow \text{Prim}_1 K$; the other cases are similar. Clearly, a pair $(\mathcal{N}_1, \mathcal{N}_2)$ with the prescribed properties will determine a cross section (the inverse of the map $r: \mathcal{N}_1 \rightarrow \text{Prim}_1 K$). Conversely, if $f: \text{Prim}_1 K \rightarrow \text{Prim}_1 \mathcal{K}$ is any cross section, then we define

$$\begin{aligned} \mathcal{N}_1 &= \{\bar{F}_\rho \in \text{Prim}_1 \mathcal{K} \mid f(r(\bar{F}_\rho)) = \bar{F}_\rho\}, \\ \mathcal{N}_2 &= \{\bar{F}_\rho \in \text{Prim}_1 \mathcal{K} \mid f(r(\bar{F}_\rho)) = \phi^*(\bar{F}_\rho)\}. \end{aligned}$$

Now we will be done if we can show that \mathcal{N}_1 and \mathcal{N}_2 are closed subsets of the (possibly non-Hausdorff) space $\text{Prim}_1 \mathcal{K}$. By assumption $\text{Prim}_1 \mathcal{K} \rightarrow \text{Prim}_1 K$ is trivial, therefore \bar{F}_ρ and $\phi^*(\bar{F}_\rho)$ can be separated by closed and open sets in $\text{Prim}_1 \mathcal{K}$; it now follows that \mathcal{N}_1 and \mathcal{N}_2 are closed. \square

We next relate C^* -structures on A to open-closed ("clopen") subsets of $\text{Prim}_1 K_1$, etc. For this purpose we define K_1 to be the split face of K which is the complement of the split face $\overline{\text{co}}(\partial_{e,o} K)$. Generally, K_1 is norm closed, but not w^* -closed. (However, K_1 is always a w^* G_δ -set by [1; Prop.II.6.5]).

Corollary 8.7. Let A be a JB-algebra whose state space K has the 3-ball property and is orientable. Then the C^* -structures on A are in 1-1 correspondence with the open and closed subsets of $\text{Prim}_1 K$ (similar statements hold for K_1 and $\partial_{e,1} K$), or what is equivalent: with the relatively w^* -closed split faces of K_1 having relatively w^* -closed complement in K_1 .

Proof. Fix any (continuous) cross section $f : \text{Prim}_1 K \rightarrow \text{Prim}_1 \mathcal{K}$. If E is an open-closed subset of $\text{Prim}_1 K$, let $f_E : \text{Prim}_1 K \rightarrow \text{Prim}_1 \mathcal{K}$ be defined by $f_E = f$ on E and $f_E = \bar{q}^* \circ f$ on $\text{Prim}_1 K \setminus E$. Then $E \mapsto f_E$ gives a 1-1 correspondence between the open-closed subsets of $\text{Prim}_1 K$ and the global orientations of K .

By Corollary 7.3 this establishes the correspondence of open-closed subsets of $\text{Prim}_1 K$ and C^* -structures on A . We leave to the reader the verification that open-closed subsets of $\text{prim}_1 K$ are in 1-1 correspondence with relatively w^* -closed split faces of K_1 having relatively w^* -closed complement in K_1 . \square

It follows from Corollary 8.7 that in the special case when $\partial_{e,0} K = \emptyset$ (i.e. when A has no one-dimensional representation), then there is a 1-1 correspondence between the C^* -structures on A and the open-closed split faces of the entire state space K . If in this case K is direct convex sum of n minimal w^* -closed split faces, then there are exactly 2^n different C^* -structures on A .

We will now show that the various \mathbb{Z}_2 -bundles need not be locally trivial.

Proposition 8.8. There exists a JB-algebra A with the 3-ball property such that none of the three associated \mathbb{Z}_2 -bundles are locally trivial.

Proof. Let H_0 be a separable infinite dimensional Hilbert space and let $H_1 = H_0 \oplus H_0$. Let K_1 be the compact operators on H_1 . Also let e_{ij} be 2×2 -matrix units in $B(H_0)$ and let $M = \text{lin}_{\mathbb{C}}\{e_{ij} | i, j = 1, 2\}$; then M is a copy of $M_2(\mathbb{C})$ imbedded in $B(H_0)$. Choose now an orthonormal basis in H_0 composed by an arbitrary orthonormal basis $\{\xi_\alpha\}$ in $e_{11}H_0$ and the corresponding basis $\{e_{21}\xi_\alpha\}$ for $e_{22}H_0 = (e_{11}H_0)^\perp$. Then the transpose map with respect to this basis will send each of the matrix units to its adjoint, i.e. $e_{ij}^t = e_{ij}^* = e_{ji}$ for $i, j = 1, 2$. Hence $M^t \subseteq M$.

Now define

$$(8.2) \quad A = \{m \oplus m^t | m = m^* \in M\} + (K_1)_{sa}.$$

Then A is a JC-algebra. (It is norm closed since K_s is norm closed and of finite codimension in A). By Corollary 3.6, A has the 3-ball property. Note also that $\text{Prim} A = \{\{0\}, (K_1)_{sa}\}$.

We will now describe the enveloping C^* -algebra \mathcal{O} of A . Let $H_2 \cong H_0 \oplus H_0$ be another replica of H_2 . Define $\psi: A \rightarrow B(H_1 \oplus H_2)$ by $\psi(a) = a \oplus a^t$, and recall (from § 5) that $\mathcal{O} = \mathcal{R}(\psi(A)) + i\mathcal{R}(\psi(A))$.

One now finds

$$(8.3) \quad \mathcal{R}(\psi(A)) = \{m \oplus m^{*t} \oplus m^{*t} \oplus m | m \in M\} + \{k \oplus k^{*t} | k \in K_1\},$$

and so

$$\mathcal{O} = \{m_1 \oplus m_2 \oplus m_2 \oplus m_1 | m_1, m_2 \in M\} + \{k_1 \oplus k_2 | k_1 \in K_1, k_2 \in K_2\}.$$

Now define

$$J_1 = \{0 \oplus m \oplus m \oplus 0 | m \in M\} + K_1 + K_2,$$

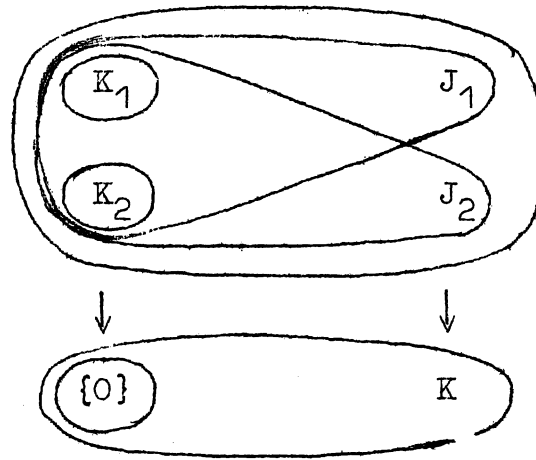
$$J_2 = \{m \oplus 0 \oplus 0 \oplus m | m \in M\} + K_1 + K_2.$$

Then J_1, J_2, K_1, K_2 are ideals in \mathcal{O} , and each is seen to be primi-

tive. Since $\text{Prim } \mathcal{O} \rightarrow \text{Prim } A$ is at most two to one, it follows that

$$(8.5) \quad \text{Prim } \mathcal{O} = \{K_1, K_2, J_1, J_2\} .$$

Note that the natural map from $\text{Prim } \mathcal{O}$ onto $\text{Prim } A$ is just $J \mapsto J \cap A$, so the ideals K_1, K_2, J_1, J_2 are sent onto $\{0\}, \{0\}, K, K$ (in this order). (As usual we identify A and $\psi(A)$). The picture below illustrates this map and the topology of the spaces involved.



Observe that the only open set in $\text{Prim } A$ containing K is all of $\text{Prim } A$; thus $\text{Prim } \mathcal{O} \rightarrow \text{Prim } A$ is locally trivial iff it is trivial. Observe also that for $i = 1, 2$ the only open sets in $\text{Prim } \mathcal{O}$ containing J_i are $\text{Prim } \mathcal{O}$ itself and $\{J_i, K_1, K_2\}$. But this shows that $\text{Prim } \mathcal{O} \rightarrow \text{Prim } A$ can not be trivial (Note $\text{Prim } \mathcal{O} = \text{Prim}_1 \mathcal{O}$ and $\text{Prim } A = \text{Prim}_1 A$, since there is no one-dimensional representation of either \mathcal{O} or A).

The same statements are seen to hold for $\hat{\mathcal{K}} \rightarrow \hat{K}$ and $\partial_e \mathcal{K} \rightarrow \partial_e K$. \square

§ 9. Orientation preserving maps.

Throughout this section \mathcal{A} and \mathcal{B} will be C^* -algebras with state spaces $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$, respectively. If $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a unital positive map, then φ^* is a w^* -continuous affine map which sends $K_{\mathcal{B}}$ into $K_{\mathcal{A}}$. Conversely, for every w^* -continuous affine map $\psi: K_{\mathcal{B}} \rightarrow K_{\mathcal{A}}$ there is a unique unital positive map $\psi_*: \mathcal{A} \rightarrow \mathcal{B}$ such that $(\psi_*)^* = \psi$ on $K_{\mathcal{B}}$. Our goal is to characterize among such maps those which correspond to $*$ -homomorphisms of \mathcal{A} into \mathcal{B} . By Proposition 2.12 we know how to characterize unital Jordan homomorphisms in terms of the dual maps of the state spaces. Thus, our principal problem is to characterize the unital $*$ -homomorphisms of \mathcal{A} into \mathcal{B} among the unital Jordan homomorphisms in terms of the dual maps between the oriented state spaces. After solving this problem (Theorem 9.2 below), we shall have little difficulty in characterizing the duals of unital $*$ -homomorphisms among all w^* -continuous affine maps between the state spaces. (Corollary 9.3).

We will need the following observations which we leave for the reader to verify: A unital positive map $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ (respectively, a w^* -continuous affine map $\varphi^*: K_{\mathcal{B}} \rightarrow K_{\mathcal{A}}$) is injective iff φ^* (respectively φ) is surjective. If $\varphi: \mathcal{A} \rightarrow B(H)$ is a unital positive map and if $\varphi(\mathcal{A})$ acts irreducibly on H , then φ^* will map the normal state space of $B(H)$ isomorphically onto some minimal split face $F_{\rho} \in \hat{K}$.

In preparation for the key definition, we introduce the notation $\mathcal{S}(\mathcal{A})$ for the set of all proper 3-balls $B(\rho, \sigma) \in K$ (with $\rho, \sigma \in \partial_e K$, $\rho \neq \sigma$ and $\sigma \in F_{\rho}$); similarly with $\mathcal{S}(\mathcal{B})$. Note that a surjective affine map of a ball $B_1 \in \mathcal{S}(\mathcal{B})$ onto a ball $B_2 \in \mathcal{S}(\mathcal{A})$ is automatically injective.

Definition. An affine map $\psi: K_{\mathcal{B}} \rightarrow K_{\mathcal{A}}$ is said to be orientation preserving if $B_1 \in \mathcal{J}(\mathcal{B})$ $B_2 \in \mathcal{J}(\mathcal{A})$ and $\psi(B_1) = B_2$ implies $\psi(\rho \times \sigma) = \psi(\rho) \times \psi(\sigma)$ for all pairs $\rho, \sigma \in \partial_e B_1$.

Thus, to say that $\psi: K_{\mathcal{B}} \times K_{\mathcal{A}}$ is orientation preserving, means that it preserves the orientation of each ball $B_1 \in \mathcal{J}(\mathcal{B})$ which is mapped onto a ball $B_2 \in \mathcal{J}(\mathcal{A})$. Note, in particular, that if \mathcal{A} (or \mathcal{B}) is abelian then $\mathcal{J}(\mathcal{A}) = \emptyset$ (respectively $\mathcal{J}(\mathcal{B}) = \emptyset$), and so every affine map $\psi: K_{\mathcal{B}} \rightarrow K_{\mathcal{A}}$ will be orientation preserving.

Lemma 9.1. Let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a unital $*$ -homomorphism. If $\rho \in \varphi^*(K_{\mathcal{B}}) \cap \partial_e K_{\mathcal{A}}$, then there exists $\tilde{\rho} \in \partial_e K_{\mathcal{B}}$ such that $\varphi^*(\tilde{\rho}) = \rho$. For each such $\tilde{\rho}$ and each $B \in \mathcal{J}(\mathcal{A})$ containing ρ , there is a unique $\tilde{B} \in \mathcal{J}(\mathcal{B})$ containing $\tilde{\rho}$ such that $\varphi^*(\tilde{B}) = B$.

Proof. The existence of $\tilde{\rho}$ follows from the Krein-Milman theorem and the observation that $(\varphi^*)^{-1}(\rho)$ is a w^* -closed face of $K_{\mathcal{B}}$.

For the remaining proof it suffices to consider the case where \mathcal{A} is a C^* -subalgebra of \mathcal{B} and φ is the canonical injection; for if $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is arbitrary then φ^* will determine an affine isomorphism of $K_{\varphi(\mathcal{A})}$ onto the split face $\varphi^*(K_{\mathcal{B}})$ of $K_{\mathcal{A}}$, so we merely have to study the canonical injection of $\varphi(\mathcal{A})$ into \mathcal{B} .

Now let $\pi: \mathcal{B} \rightarrow B(H)$ be the GNS-representation associated with $\tilde{\rho}$, and let $\xi \in H$ be the cyclic vector. Since $\rho = \tilde{\rho}|_{\mathcal{A}}$ is a pure state of $K_{\mathcal{A}}$, then $\pi(\mathcal{A})$ will act irreducibly on $(\pi(\mathcal{A})\xi)^{\perp}$.

Let $Q \in B(H)$ be the projection onto $(\pi(\mathcal{A})\xi)^{\perp}$, and for each $T \in B(H)$ let $\pi_Q(T)$ denote QTQ considered as an operator on QH ; thus $\pi_Q: B(H) \rightarrow B(QH)$. Note that $Q \in \pi(\mathcal{A})'$, so $\pi_Q \circ \pi \circ \varphi$ is an irreducible representation of \mathcal{A} on QH . Thus, $(\pi_Q \circ \pi \circ \varphi)^*$ will

map the normal state space $\mathcal{N}(B(QH))$ of $B(QH)$ onto $F_\rho \subseteq K_{\mathcal{O}}$. Note that π_Q^* maps $\mathcal{N}(B(QH))$ isomorphically onto the norm closed face F_Q of $\mathcal{N}(B(H))$ given by

$$(9.1) \quad F_Q = \{\omega \in \mathcal{N}(B(QH)) \mid \langle Q, \omega \rangle = 1\},$$

and π^* maps $\mathcal{N}(B(H))$ isomorphically onto $F_{\tilde{\rho}} \subseteq K_{\mathcal{B}}$. It follows that the restriction map φ^* sends the face $G = \pi^*(F_Q) \subseteq F_{\tilde{\rho}}$ isomorphically onto F_ρ . Therefore G must contain a ball $\tilde{B} \in \mathcal{S}(\mathcal{B})$ such that $\tilde{\rho} \in \tilde{B}$ and $\varphi^*(\tilde{B}) = B$.

To establish uniqueness of \tilde{B} , we consider an arbitrary ball $B' \in \mathcal{S}(\mathcal{B})$ such that $\tilde{\rho} \in B'$ and $\varphi^*(B') = B$. To show $B' = \tilde{B}$, it suffices to show that B' is contained in the face $G = \pi^*(F_Q)$ defined above.

Suppose, for contradiction, that this is not the case. Then we can find $\sigma \in \partial_{\mathcal{B}} B' \setminus G$. Since $\sigma \in B' = B(\sigma, \tilde{\rho}) \subseteq F_{\tilde{\rho}}$, there will exist a unit vector $\eta \in H$ such that $\sigma = \omega_\eta \circ \pi$ (where, as usual, ω_η denotes the vector state determined by η). Since $\sigma \notin G$, then $\omega_\eta \notin F_Q$, so $\eta \notin QH$. We are going to show that this will contradict the hypothesis $\varphi^*(B') = B$.

Since π^* is an isomorphism from $\mathcal{N}(B(H))$ onto $F_{\tilde{\rho}}$, the pre-image of $B' = B(\tilde{\rho}, \sigma)$ will be $B(\omega_{\xi}, \omega_\eta)$. It follows that $(\pi \circ \varphi)^*$ maps $B(\omega_{\xi}, \omega_\eta)$ onto B . If P is the projection of H onto $\text{lin}\{\xi, \eta\}$, then we can identify $B(\omega_{\xi}, \omega_\eta)$ with the state space of $B(PH) \cong PB(H)P$; thus the extreme points of $B(\omega_{\xi}, \omega_\eta)$ are the vector states ω_γ where

$$\gamma = \|s\xi + t\eta\|^{-1}(s\xi + t\eta); \quad s, t \in \mathbb{C}.$$

Since $Q \in \pi(\mathcal{O})'$, we have for each $a \in \mathcal{O}$:

$$(\pi(a)\gamma | \gamma) = (\pi(a)Q\gamma | Q\gamma) + (\pi(a)(I-Q)\gamma | (I-Q)\gamma).$$

Let $\delta = \|(I-Q)\gamma\|^{-1}(I-Q)\gamma$. (Note $(I-Q)\gamma \neq 0$ if $t \neq 0$, so $\gamma \notin QH$). Then on $\pi(\mathcal{O}\mathcal{L})$ the state ω_γ will be a convex combination of ω_δ and other vector states for each $t \neq 0$. It follows that $(\pi \circ \varphi)^* \omega_\gamma$ is not pure on $\mathcal{O}\mathcal{L}$ unless it equals ω_δ or ω_ξ . Hence $(\pi \circ \varphi)^*$ can not map $B(\omega_\xi, \omega_\eta)$ onto B , and therefore φ^* can not map $B' = B(\tilde{\rho}, \sigma)$ onto B . This is the desired contradiction. \square

Theorem 9.2. Let $\varphi: \mathcal{O}\mathcal{L} \rightarrow \mathcal{O}\mathcal{B}$ be a unital Jordan homomorphism from a C^* -algebra $\mathcal{O}\mathcal{L}$ into a C^* -algebra $\mathcal{O}\mathcal{B}$. Then φ is a $*$ -homomorphism iff $\varphi^*: K_{\mathcal{O}\mathcal{B}} \rightarrow K_{\mathcal{O}\mathcal{L}}$ is an orientation preserving map between the state spaces of $\mathcal{O}\mathcal{L}$ and $\mathcal{O}\mathcal{B}$.

Proof. 1.) Assume first that φ is a $*$ -homomorphism, and let $B_1 \in \mathcal{J}(\mathcal{O}\mathcal{B})$ and $B_2 \in \mathcal{J}(\mathcal{O}\mathcal{L})$ satisfy $\varphi^*(B_1) = B_2$. Choose $\rho \in \partial_e B_2$ and $\tilde{\rho} \in \partial_e B_1$ with $\varphi^* \tilde{\rho} = \rho$. As in the proof of Lemma 9.1, let $\pi: \mathcal{O}\mathcal{B} \rightarrow B(H)$ be the GNS-representation associated with $\tilde{\rho}$, ξ the corresponding cyclic vector, $Q \in B(H)$ the projection onto $(\pi(\varphi(\mathcal{O}\mathcal{L}))\xi)^\perp$, and F_Q the face of $\mathcal{N}(B(H))$ defined by (9.1). Again $\pi_Q \circ \pi \circ \varphi: \mathcal{O}\mathcal{L} \rightarrow B(QH)$ is an irreducible representation of $\mathcal{O}\mathcal{L}$. Now π_Q^* maps $\mathcal{N}(B(QH))$ isomorphically onto $F_Q \subseteq (B(H))$, and it is easily seen from the definition of orientation that π_Q^* is an orientation preserving map from $\mathcal{N}(B(QH))$ to $\mathcal{N}(B(H))$. (We may view $\mathcal{N}(B(QH))$ and $\mathcal{N}(B(H))$ as minimal split faces of the state spaces of $B(QH)$ and $B(H)$ respectively, and with the inherited orientations). Let $\tilde{\pi}: \mathcal{O}\mathcal{B}^{**} \rightarrow B(H)$ be the σ -weakly continuous extension of π , and note that $\tilde{\pi}(\mathcal{O}\mathcal{B}^{**}) = B(H)$. Thus for some central projection $c \in \mathcal{O}\mathcal{B}^{**}$, $\tilde{\pi}$ will be an isomorphism on $c \mathcal{O}\mathcal{B}^{**}$ and $\tilde{\pi}$ will vanish on $(I-c) \mathcal{O}\mathcal{B}^{**}$. From this it follows that $\pi^*: \mathcal{N}(B(H)) \rightarrow K_{\mathcal{O}\mathcal{B}}$ is orientation preserving. The same argument shows that $(\pi_Q \circ \pi \circ \varphi)^*$ is an orientation preserving isomorphism of $\mathcal{N}(B(QH))$ onto $F_\rho \subseteq K$.

From this it follows that $\varphi^* : \pi^*(F_Q) \rightarrow F_\rho$ is orientation preserving; since $B_1 \subseteq \pi^*(F_Q)$ (by the uniqueness statement of Lemma 9.1), this shows that $\varphi^* : K_\beta \rightarrow K_{\mathcal{O}\mathcal{L}}$ is orientation preserving.

2.) Assume next that $\varphi^* : K_\beta \rightarrow K_{\mathcal{O}\mathcal{L}}$ is orientation preserving; we will show that φ is a $*$ -homomorphism. Let \mathcal{E} be the C^* -algebra generated by $\varphi(\mathcal{O}\mathcal{L})$ in β ; clearly it suffices to show that $\varphi : \mathcal{O}\mathcal{L} \rightarrow \mathcal{E}$ is a $*$ -homomorphism.

We will first show that $\varphi^* : K_\mathcal{E} \rightarrow K_{\mathcal{O}\mathcal{L}}$ is orientation preserving. If $B_1 \in \mathcal{J}(\mathcal{E})$, $B_2 \in \mathcal{J}(\mathcal{O}\mathcal{L})$, and $\varphi^*(B_1) = B_2$, then we choose a ball $B_0 \in \mathcal{J}(\beta)$ such that the restriction map from K_β to $K_\mathcal{E}$ is a surjection of B_0 onto B_1 . (This is possible by Lemma 9.1). By the first part of the proof, this restriction map is orientation preserving. By hypothesis, its composition with φ^* , i.e. $\rho \mapsto \varphi^*(\rho|_{\mathcal{E}}) = \varphi^*(\rho)$, will also preserve orientations; this shows that $\varphi^* : K_\mathcal{E} \rightarrow K_{\mathcal{O}\mathcal{L}}$ is orientation preserving.

Now let $\pi : \mathcal{E} \rightarrow B(H)$ be any irreducible $*$ -representation of \mathcal{E} . Since $\varphi(\mathcal{O}\mathcal{L})$ generates \mathcal{E} , $\pi \circ \varphi : \mathcal{O}\mathcal{L} \rightarrow B(H)$ will be an irreducible Jordan representation of $\mathcal{O}\mathcal{L}$. By [25; Cor.3.4] $\pi \circ \varphi$ is either a $*$ -homomorphism or a $*$ -anti-homomorphism. Note that π^* will be an orientation preserving isomorphism from $\mathcal{N}(B(H))$ onto F_ρ for some $\rho \in \partial_e K$. Since φ^* is orientation preserving, the map $(\pi \circ \varphi)^* : \mathcal{N}(B(H)) \rightarrow K$ will also be. But this rules out the case where $\pi \circ \varphi$ is a $*$ -anti-homomorphism with $\dim \pi(\varphi(\mathcal{O}\mathcal{L})) > 1$, for then $(\pi \circ \varphi)^*$ would reverse the orientation of each ball in $\mathcal{J}(B(H))$. Hence for all $a, b \in \mathcal{O}\mathcal{L}$:

$$\pi(\varphi(ab)) = \pi(\varphi(a))\pi(\varphi(b)) = \pi(\varphi(a)\varphi(b)).$$

Since π is an arbitrary irreducible representation of \mathcal{E} , this shows that $\varphi(ab) = \varphi(a)\varphi(b)$. \square

Recall from [17] or [31] that the norm closed faces of the state space $K_{\mathcal{O}\mathcal{L}}$ of a C^* -algebra $\mathcal{O}\mathcal{L}$ are norm exposed; in fact they are precisely the sets

$$(9.2) \quad F_q = \{\omega \in K_{\mathcal{O}\mathcal{L}} \mid \langle q, \omega \rangle = 1\},$$

where q is a projection in $\mathcal{O}\mathcal{L}^{**}$. Thus, such faces occur in pairs F_q, F_{I-q} of "quasicomplementary projective faces" which were characterized abstractly in terms of the geometry of the convex set $K_{\mathcal{O}\mathcal{L}}$ in [4]. (In particular, F_q is said to be the "quasicomplement" of F_{I-q} , and vice versa).

Corollary 9.3. Let $\varphi: \mathcal{O}\mathcal{L} \rightarrow \mathcal{O}\mathcal{B}$ be a unital positive map between the C^* -algebras $\mathcal{O}\mathcal{L}, \mathcal{O}\mathcal{B}$. Then φ is a $*$ -homomorphism iff $\varphi^*: K_{\mathcal{O}\mathcal{B}} \rightarrow K_{\mathcal{O}\mathcal{L}}$ is an orientation preserving map and $(\varphi^*)^{-1}$ preserves quasicomplements.

Proof. The corollary follows by combining Theorem 9.2 with Proposition 2.12 where the Jordan homomorphisms between JB-algebras were characterized among all unital positive maps. \square

This corollary shows that the category of C^* -algebras with unital $*$ -homomorphisms and the category of compact convex sets satisfying the conditions of Corollary 7.2 with orientation preserving maps, are equivalent.

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